# A Short Tutorial on Game Theory

*EE228a, Fall 2002*
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## Outline

- Introduction
- Complete-Information Strategic Games
  - Static Games
  - Repeated Games
  - Stackelberg Games
- Cooperative Games
  - Bargaining Problem
  - Coalitions



## Outline

- Introduction
  - What is game theory about?
  - Relevance to networking research
  - Elements of a game
- Non-Cooperative Games
  - Static Complete-Information Games
  - Repeated Complete-Information Games
  - Stackelberg Games
- Cooperative Games
  - Nash's Bargaining Solution
  - Coalition: the Shapley Value



## What Is Game Theory About?

- To understand how decision-makers interact
- A brief history
  - 1920s: study on strict competitions
  - 1944: Von Neumann and Morgenstern's book
    *Theory of Games and Economic Behavior*
  - After 1950s: widely used in economics, politics, biology…
    - Competition between firms
    - Auction design
    - Role of punishment in law enforcement
    - International policies
    - Evolution of species



## Relevance to Networking Research

- Economic issues becomes increasingly important
  - Interactions between human users
    - congestion control
    - resource allocation
  - Independent service providers
    - Bandwidth trading
    - Peering agreements
- Tool for system design
  - Distributed algorithms
  - Multi-objective optimization
  - Incentive compatible protocols



## Elements of a Game: Strategies

- Decision-maker's choice(s) in any given situation
- Fully known to the decision-maker
- Examples
  - Price set by a firm
  - Bids in an auction
  - Routing decision by a routing algorithm
- Strategy space: set of all possible actions
  - Finite *vs* infinite strategy space
- Pure *vs* mixed strategies
  - Pure: deterministic actions
  - Mixed: randomized actions

## Elements of a Game: Preference and Payoff

- Preference
  - Transitive ordering among strategies
    if $a >> b$, $b >> c$, then $a >> c$
- Payoff
  - An order-preserving mapping from preference to $R^+$
  - Example: in flow control, $U(x) = log(1+x) - px$
    payoff ↑ (for $U(x)$), action ↑ (for $x$)



## Rational Choice

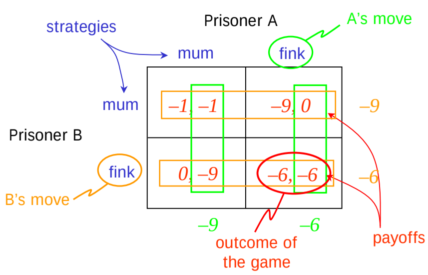
- Two axiomatic assumptions on games

1. *In any given situation a decision-maker always chooses the action which is the best according to his/her preferences (a.k.a. rational play).*
2. *Rational play is common knowledge among all players in the game.*

Question: Are these assumptions reasonable?



## Example: Prisoners' Dilemma





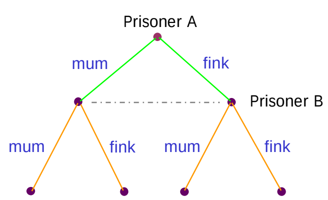

## Different Types of Games

- Static *vs* multi-stage
  - Static: game is played only once
    - Prisoners' dilemma
  - Multi-stage: game is played in multiple rounds
    - Multi-round auctions, chess games
- Complete *vs* incomplete information
  - Complete info.: players know each others' payoffs
    - Prisoners' dilemma
  - Incomplete info.: other players' payoffs are not known
    - Sealed auctions



## Representations of a Game

- Normal- *vs* extensive-form representation
  - Normal-form
    - like the one used in previous example
  - Extensive-form



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  - Stackelberg Games
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  - Nash's Bargaining Problem
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## Static Games

- Model
  - Players know each others' payoffs
  - But do not know which strategies they would choose
  - Players simultaneously choose their strategies
    - ⇒ Game is over and players receive payoffs based on the combination of strategies just chosen
- Question of Interest:
  - What outcome would be produced by such a game?



## Example: Cournot's Model of Duopoly

- Model (from Gibbons)
  - Two firms producing the same kind of product in quantities of $q_1$ and $q_2$, respectively
  - Market clearing price $p=A-q_1-q_2$
  - Cost of production is $C$ for both firms
  - Profit for firm $i$

$$J_i = p_i q_i - C q_i = (A - q_1 - q_2)\, q_i - C q_i$$
$$= (A - C - q_1 - q_2)\, q_i$$

define $B \equiv A - C$

  - Objective: choose $q_i$ to maximize profit

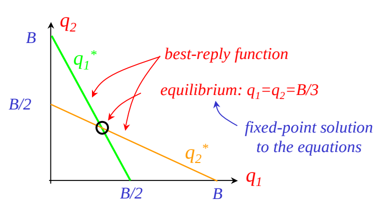
$$q_i^* = argmax_{qi}\, (B - q_1 - q_2)\, q_i$$



## A Simple Example: Solution

- Firm $i$'s best choice, given its competitor's $q$

$$\begin{cases} q_1^* = (B - q_2)/2 \\ q_2^* = (B - q_1)/2 \end{cases}$$






## Solution to Static Games

- Nash Equilibrium (*J. F. Nash, 1950*)
  - Mathematically, a strategy profile $(s_1^*, \ldots, s_i^*, \ldots, s_n^*)$ is a Nash Equilibrium if for each player $i$

$$U_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*)$$
$$\geq U_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*),$$

  for each feasible strategy $s_i$
  - Plain English: a situation in which no player has incentive to deviate
  - It's fixed-point solution to the following system of equations

$$s_i = argmax_s\, U_i(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n), \ \forall i$$

- Other solution concepts (see references)



## An Example on Mixed Strategies

- Pure-Strategy Nash Equilibrium may not exist

Player A

| Player B | | Head (H) | Tail (T) |
|---|---|---|---|
| | H | 1, –1 | –1, 1 |
| | T | –1, 1 | 1, –1 |

Cause: each player tries to outguess his opponent!



## Example: Best Reply

- Mixed Strategies
  - Randomized actions to avoid being outguessed
- Players' strategies and expected payoffs
  - Players play H w.p. $p$ and play T w.p. $1-p$
  - Expected payoff of Player A

$$p_a p_b + (1-p_a)(1-p_b) - p_a(1-p_b) - p_b(1-p_a)$$
$$= (1 - 2p_b) + p_a(4p_b - 2)$$

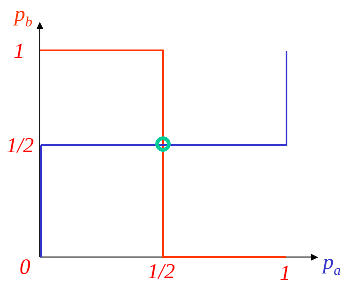
*So …*

*if* $p_b > 1/2$, $p_a^* = 1$ *(i.e. play H);*

*if* $p_b > 1/2$, $p_a^* = 0$ *(i.e. play T);*

*if* $p_b = 1/2$, *then playing either H or T is equally good*

## Example: Nash Equilibrium






## Existence of Nash Equilibrium

- Finite strategy space (*J. F. Nash, 1950*)

  *A n-player game has at least one Nash equilibrium, possibly involving mixed strategy.*

- Infinite strategy space (*R.B. Rosen, 1965*)

  *A pure-strategy Nash Equilibrium exists in a n-player concave game.*

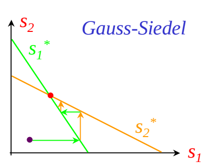
  *If the payoff functions satisfy diagonally strict concavity condition, then the equilibrium is unique.*

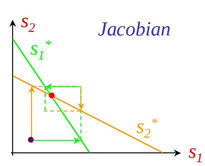
  $(\underline{s}_1 - \underline{s}_2)\,[\, r_j \nabla J_j(\underline{s}_1)\,] + (\underline{s}_2 - \underline{s}_1)\,[\, r_j \nabla J_j(\underline{s}_2)\,] < 0$



## Distributed Computation of Nash Equilibrium

- Nash equilibrium as result of "learning"
  - Players iteratively adjust their strategies based on locally available information
  - Equilibrium is reached if there is a steady state
- Two commonly used schemes

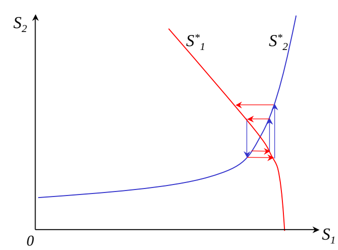





## Convergence of Distributed Algorithms

- Algorithms may not converge for some cases






## Suggested Readings

- J.F. Nash. "*Equilibrium Points in N-Person Games.*" Proc. of National Academy of Sciences, vol. 36, 1950.
  - *A "must-read" classic paper*
- R.B. Rosen. "*Existence and Uniqueness of Equilibrium Points for Concave N-Person Games.*" Econometrica, vol. 33, 1965.
  - *Has many useful techniques*
- A. Orda et al. "*Competitive Routing in Multi-User Communication Networks.*" IEEE/ACM Transactions on Networking, vol. 1, 1993.
  - *Applies game theory to routing*
- And many more…



## Multi-Stage Games

- General model
  - Game is played in multiple rounds
    - Finite or infinitely many times
  - Different games could be played in different rounds
    - Different set of actions or even players
  - Different solution concepts from those in static games
    - Analogy: optimization *vs* dynamic programming
- Two special classes
  - Infinitely repeated games
  - Stackelberg games

## Infinitely Repeated Games

- Model
  - A single-stage game is repeated infinitely many times
  - Accumulated payoff for a player

$$J=\tau_1+\delta\tau_2+\ldots+\delta^{n-1}\tau_n+\ldots=\Sigma_i\,\delta^{i-1}\tau_i$$

*discount factor* *payoff from stage n*

- Main theme: play socially more efficient moves
  - Everyone promises to play a socially efficient move in each stage
  - Punishment is used to deter "cheating"
  - Example: justice system



## Cournot's Game Revisited. I

- Cournot's Model
  - At equilibrium each firm produces $B/3$, making a profit of $B^2/9$
  - Not an "ideal" arrangement for either firm, because…

    *If a central agency decides on production quantity $q_m$*

    $q_m = \text{argmax } (B - q)\, q = B/2$

    *so each firm should produce $B/4$ and make a profit of $B^2/8$*
  - An aside: why $B/4$ is not played in the static game?

    *If firm A produces $B/4$, it is more profitable for firm B to produce $3B/8$ than $B/4$*

    *Firm A then in turn produces $5B/16$, and so on…*



## Cournot's Game Revisited. II

- Collaboration instead of competition

  *Q: Is it possible for two firms to reach an agreement to produce $B/4$ instead of $B/3$ each?*

  *A: That would depend on how important future return is to each firm…*

  *A firm has two choices in each round:*
  - *Cooperate: produce $B/4$ and make profit $B^2/8$*
  - *Cheat: produce $3B/8$ and make profit $9B^2/64$*

    *But in the subsequent rounds, cheating will cause*

    *– its competitor to produce $B/3$ as punishment*

    *– its own profit to drop back to $B^2/9$*



## Cournot's Game Revisited. III

- Is there any incentive for a firm not to cheat?

  *Let's look at the accumulated payoffs:*
  - *If it cooperates:*

    $S_c = (1+\delta+\delta^2+\delta^3+\ldots)\, B^2/8 = B^2/8(1-\delta)$
  - *If it cheats:*

    $S_d = 9B^2/64 + (\delta+\delta^2+\delta^3+\ldots)\, B^2/9$

    $= \{9/64 + \delta/9(1-\delta)\}\, B^2$

  *So it will not cheat if $S_c > S_d$. This happens only if $\delta > 9/17$.*
- Conclusion
  - If future return is valuable enough to each player, then strategies exist for them to play socially efficient moves.
- Question: What happens if player cheats in a later round?



## Strategies in Repeated Games

- A strategy
  - is no longer a single action
  - but a complete plan of actions
  - based on possible history of plays up to current stage
  - usually includes some punishment mechanism
  - Example: in Cournot's game, a player's strategy is

    *Produce $B/4$ in the first stage. In the $n^{th}$ stage, produce $B/4$ if both firms have produced $B/4$ in each of the $n$–1 previous stages; otherwise, produce $B/3$.*

*history* *punishment*



## Equilibrium in Repeated Games

- Subgame-perfect Nash equilibrium (SPNE)
  - A subgame starting at stage $n$ is
    - *identical to the original infinite game*
    - *associated with a particular sequence of plays from the first stage to stage $n$–1*
  - A SPNE constitutes a Nash equilibrium in every subgame
- Why subgame perfect?
  - It is all about creditable threats:

    *Players believe the claimed punishments indeed will be carried out by others, when it needs to be evoked.*
  - So a creditable threat has to be a Nash equilibrium for the subgame.

## Known Results for Repeated Games

- Friedman's Theorem (1971)

  *Let G be a single-stage game and $(e_1,\ldots, e_n)$ denote the payoff from a Nash equilibrium of G.*

  *If $\underline{x}=(x_1, \ldots, x_n)$ is a feasible payoff from G such that $x_i \geq e_i, \forall i$, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game of G which achieves $\underline{x}$, provided that discount factor $\delta$ is close enough to one.*

*Assignment*:

*Apply this theorem to Cournot's game on an agreement other than B/4.*



## Suggested Readings

- J. Friedman. *"A Non-cooperative Equilibrium for Super-games."* Review of Economic Studies, vol. 38, 1971.
  - *Friedman's original paper*
- R. J. La and V. Anantharam. *"Optimal Routing Control: Repeated Game Approach,"* IEEE Transactions on Automatic Control, March 2002.
  - *Applies repeated game to improve the efficiency of competitive routing*



## Stackelberg Games

- Model
  - One player (leader) has dominant influence over another
  - Typically there are two stages
  - One player (leader) moves first
  - Then the other follows in the second stage
  - Can be generalized to have
    - multiple groups of players
    - Static games in both stages
- Main Theme
  - Leader plays by backwards induction, based on the anticipated behavior of his/her follower.



## Stackelberg's Model of Duopoly

- Assumptions
  - Firm 1 chooses a quantity $q_1$ to produce
  - Firm 2 observes $q_1$ and then chooses a quantity $q_2$
- Outcome of the game
  - For any given $q_1$, the best move for Firm 2 is

    $$q_2^* = (B - q_1)/2$$
  - Knowing this, Firm 1 chooses $q_1$ to maximize

    $$J_1 = (B - q_1 - q_2^*)\, q_1 = q_1(B - q_1)/2$$

    which yields

    $q_1^* = B/2$, and $q_2^* = B/4$

    $J_1^* = B^2/8$, and $J_2^* = B^2/16$



## Suggested Readings

- Y. A. Korilis, A. A. Lazar and A. Orda. " *Achieving Network Optima Using Stackelberg Routing Strategies."* IEEE/ACM Trans on Networking, vol.5, 1997.
  - *Network leads users to reach system optimal equilibrium in competitive routing.*
- T. Basar and R. Srikant. *"Revenue Maximizing Pricing and Capacity Expansion in a Many-User Regime."* INFOCOM 2002, New York.
  - *Network charges users price to maximize its revenue.*



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  - Nash's Bargaining Problem
  - Coalitions: the Shapley value

## Cooperation In Games

- Incentive to cooperate
  - Static games often lead to inefficient equilibrium
  - Achieve more efficient outcomes by acting together
    - Collusion, binding contract, side payment...
- Pareto Efficiency

  *A solution is Pareto efficient if there is no other feasible solution in which some player is better off and no player is worse off.*

  - Pareto efficiency may be neither socially optimal nor fair
  - Socially optimal $\Rightarrow$ Pareto efficient
  - Fairness issues
    - Reading assignment as an example



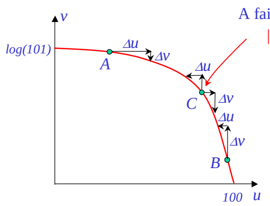
## Nash's Bargaining Problem

- Model
  - Two players with interdependent payoffs $U$ and $V$
  - Acting together they can achieve a set of feasible payoffs
  - The more one player gets, the less the other is able to get
  - And there are multiple Pareto efficient payoffs
- Q: which feasible payoff would they settle on?
  - Fairness issue
- Example (from Owen):
  - Two men try to decide how to split *\$100*
  - One is very rich, so that $U(x) \cong x$
  - The other has only *\$1*, so $V(x) \cong \log(1+x) - \log 1 = \log(1+x)$
  - How would they split the money?



## Intuition

- Feasible set of payoffs
  - Denote $x$ the amount that the rich man gets
  - $(u,v)=(x, \log(101-x)),\ x \in [0,100]$




A fair split should satisfy

$| \Delta u/u | = | \Delta v/v |$

Let $\Delta \to 0$, $du/u = -dv/v$

Or $du/u + dv/v = 0$, or

$vdu+udv=0$, or $d(uv)=0$.

$\Rightarrow$ Find the allocation which maximizes $U \times V$

$\Rightarrow x^*=76.8$!



## Nash's Axiomatic Approach (1950)

- A solution $(u^*,v^*)$ should be
  - Rational
    - $(u^*,v^*) \geq (u_0,v_0)$, where $(u_0,v_0)$ is the worst payoffs that the players can get.
  - Feasible
    - $(u^*,v^*) \in S$, the set of feasible payoffs.
  - Pareto efficient
  - Symmetric
    - If $S$ is such that $(u,v) \in S \Leftrightarrow (v,u) \in S$, then $u^*=v^*$.
  - Independent from linear transformations
  - Independent from irrelevant alternatives
    - Suppose $T \subset S$. If $(u^*,v^*) \in T$ is a solution to $S$, then $(u^*,v^*)$ should also be a solution to $T$.



## Results

- There is a <u>unique</u> solution which
  - satisfies the above axioms
  - maximizes the product of two players' additional payoffs

    $(u-u_0)(v-v_0)$

- This solution can be enforced by "threats"
  - Each player independently announces his/her threat
  - Players then bargain on their threats
  - If they reach an agreement, that agreement takes effect;
  - Otherwise, initially announced threats will be used
- Different fairness criteria can be achieved by changing the last axiom (see references)



## Suggested Readings

- J. F. Nash. *"The Bargaining Problem."* Econometrica, vol.18, 1950.
  - *Nash's original paper. Very well written.*
- X. Cao. *"Preference Functions and Bargaining Solutions."* Proc. of the 21th CDC, NYC, NY, 1982.
  - *A paper which unifies all bargaining solutions into a single framework*
- Z. Dziong and L.G. Mason. *"Fair–Efficient Call Admission Control Policies for Broadband Networks – a Game Theoretic Framework,"* IEEE/ACM Trans. On Networking, vol.4, 1996.
  - *Applies Nash's bargaining solution to resource allocation problem in admission control (multi-objective optimization)*

## Coalitions

- Model
  - Players (n>2) $N$ form coalitions among themselves
  - A coalition is any nonempty subset of $N$
  - Characteristic function $V$ defines a game

    $V(S)$=payoff to $S$ in the game between $S$ and $N$-$S$, $\forall S \subset N$

    $V(N)$=total payoff achieved by all players acting together

    $V(\cdot)$ is assumed to be super-additive

    $\forall S, T \subset N, V(S+T) \geq V(S)+V(T)$
- Questions of Interest
  - Condition for forming stable coalitions
  - When will a single coalition be formed?
    - How to distribute payoffs among players in a fair way?



## Core Sets

- Allocation $X=(x_1, \ldots, x_n)$

  $x_i \geq V(\{i\}), \forall i \in N; \quad \Sigma_{i \in N} x_i = V(N).$
- The core of a game

  a set of allocation which satisfies $\Sigma_{i \in S} x_i \geq V(S), \forall S \subset N$

  $\Rightarrow$ If the core is nonempty, a single coalition can be formed
- An example
  - A Berkeley landlord (L) is renting out a room
  - Al (A) and Bob (B) are willing to rent the room at \$600 and \$800, respectively
  - Who should get the room at what rent?



## Example: Core Set

- Characteristic function of the game
  - $V(L)=V(A)=V(B)=V(A+B)=0$
  - Coalition between $L$ and $A$ or $L$ and $B$

    If rent = $x$, then $L$'s payoff = $x$, $A$'s payoff = $600 - x$

    so $V(L+A)=600$. Similarly, $V(L+B)=800$
  - Coalition among $L$, $A$ and $B$: $V(L+A+B)=800$
- The core of the game

$$\begin{cases} x_L + x_A \geq 600 \\ x_L + x_B \geq 800 \\ x_L + x_A + x_B = 800 \end{cases} \Rightarrow core=\{(y, 0, 800-y), 600 \leq y \leq 800\}$$



## Fair Allocation: the Shapley Value

- Define solution for player $i$ in game $V$ by $P_i(V)$
- Shapley's axioms
  - $P_i$'s are independent from permutation of labels
  - Additive: if $U$ and $V$ are any two games, then

    $P_i(U+V) = P_i(U) + P_i(V), \forall i \in N$
  - $T$ is a carrier of $N$ if $V(S \cap T)=V(S), \forall S \subset N$. Then for any carrier $T$, $\Sigma_{i \in T} P_i = V(T)$.
- Unique solution: Shapley's value (1953)

$$P_i = \Sigma_{S \subset N} \frac{(|S|-1)!\,(N-|S|)!}{N!} [V(S) - V(S - \{i\})]$$

- Intuition: a probabilistic interpretation



## Suggested Readings

- L. S. Shapley. "*A Value for N-Person Games.*" Contributions to the Theory of Games, vol.2, Princeton Univ. Press, 1953.
  - *Shapley's original paper.*
- P. Linhart *et al.* "*The Allocation of Value for Jointly Provided Services.*" Telecommunication Systems, vol. 4, 1995.
  - *Applies Shapley's value to caller-ID service.*
- R. J. Gibbons *et al.* "*Coalitions in the International Network.*" Tele-traffic and Data Traffic, ITC-13, 1991.
  - *How coalition could improve the revenue of international telephone carriers.*



## Summary

- Models
  - Strategic games
    - Static games, multi-stage games
  - Cooperative games
    - Bargaining problem, coalitions
- Solution concepts
  - Strategic games
    - Nash equilibrium, Subgame-perfect Nash equilibrium
  - Cooperative games
    - Nash's solution, Shapley value
- Application to networking research
  - Modeling and design

## References


- R. Gibbons, "*Game Theory for Applied Economists,*" Princeton Univ. Press, 1992.
  - *an easy-to-read introductory to the subject*
- M. Osborne and A. Rubinstein, "*A Course in Game Theory,*" MIT Press, 1994.
  - *a concise but rigorous treatment on the subject*
- G. Owen, "*Game Theory,*" Academic Press, 3rd ed., 1995.
  - *a good reference on cooperative games*
- D. Fudenberg and J. Tirole, "*Game Theory,*" MIT Press, 1991.
  - *a complete handbook; "the bible for game theory"*
  - *http://www.netlibrary.com/summary.asp?id=11352*