Brief Detour into Random Processes
Figure 1.1 An ensemble of sample functions:
\[ \{x_j(t) \mid j = 1,2,\ldots,n\} \]
S → X(t,s) -T ≤ t ≤ T  \hspace{2cm} (1.1)

2T: The total observation interval

s_j → X(t,s_j) = x_j(t)  \hspace{2cm} (1.2)

x_j(t) = \text{sample function}

At t = t_k, x_j(t_k) is a random variable (RV).

To simplify the notation, let X(t,s) = X(t)

X(t): Random process, an ensemble of time function together with a probability rule.

**Difference between RV and RP**

RV: The outcome is mapped into a number

RP: The outcome is mapped into a function of time
1.3 Stationary Process

Stationary Process:

The statistical characterization of a process is independent of the time at which observation of the process is initiated.

Nonstationary Process:

Not a stationary process (unstable phenomenon)

Consider $X(t)$ which is initiated at $t = -\infty$,

$X(t_1), X(t_2), \ldots, X(t_k)$ denote the RV obtained at $t_1, t_2, \ldots, t_k$

For the RP to be stationary in the strict sense (strictly stationary)

The joint distribution function

$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k)$  \hspace{1cm} (1.3)

For all time shift $\tau$, all $k$, and all possible choice of $t_1, t_2, \ldots, t_k$
1.4 Mean, Correlation, and Covariance Function

Let $X(t)$ be a strictly stationary RP

The mean of $X(t)$ is

$$
\mu_X(t) = E[X(t)]
= \int_{-\infty}^{\infty} x f_{X(t)}(x) \, dx
= \mu_X \quad \text{for all } t
$$

$f_{X(t)}(x)$: the first order pdf.

The autocorrelation function of $X(t)$ is

$$
R_X(t_1,t_2) = E[X(t_1)X(t_2)]
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1,x_2) \, dx_1 \, dx_2
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(0)X(t_2-t_1)}(x_1,x_2) \, dx_1 \, dx_2
= R_X(t_2-t_1) \quad \text{for all } t_1 \text{ and } t_2
$$
Properties of the autocorrelation function

For convenience of notation, we redefine

\[ R_X(\tau) = E[X(t - \tau)X(t)], \quad \text{for all } t \]  \hspace{1cm} (1.11)

1. The mean-square value

\[ R_X(0) = E[X^2(t)], \quad \tau = 0 \]  \hspace{1cm} (1.12)

2. \[ R_X(\tau) = R(-\tau) \]  \hspace{1cm} (1.13)

3. \[ |R_X(\tau)| \leq R_X(0) \]  \hspace{1cm} (1.14)
The $R_X(\tau)$ provides the interdependence information of two random variables obtained from $X(t)$ at times $\tau$ seconds apart.
Example 1.2 \( X(t) = A \cos(2\pi f_c t + \Theta) \) (1.15)

\[
f_{\Theta}(\theta) = \begin{cases} 
\frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\
0, & \text{elsewhere}
\end{cases}
\] (1.16)

\[
R_X(\tau) = E[X(t+\tau)X(t)] = \frac{A^2}{2} \cos(2\pi f_c t)
\] (1.17)
1.6 Transmission of a random Process Through a Linear Time-Invariant Filter (System)

\[ Y(t) = \int_{-\infty}^{\infty} h(\tau_1)X(t - \tau_1) \, d\tau_1 \]

where \( h(t) \) is the impulse response of the system

\[ \mu_Y(t) = E[Y(t)] \]

\[ = E \left[ \int_{-\infty}^{\infty} h(\tau_1)X(t - \tau_1) \, d\tau_1 \right] \]  \hspace{1cm} (1.27)

If \( E[X(t)] \) is finite

\[ = \int_{-\infty}^{\infty} h(\tau_1)E[X(t - \tau_1)] \, d\tau_1 \]

and system is stable

\[ = \int_{-\infty}^{\infty} h(\tau_1)\mu_X(t - \tau_1) \, d\tau_1 \]  \hspace{1cm} (1.28)

If \( X(t) \) is stationary, \( \mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau_1) \, d\tau_1 = \mu_X H(0) \)

\( H(0) \): System DC response.
Consider autocorrelation function of $Y(t)$:

$$R_Y(t, \mu) = E[Y(t)Y(\mu)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(\mu-\tau_2)\,d\tau_2\right] \quad (1.30)$$

If $E[X^2(t)]$ is finite and the system is stable,

$$R_Y(t, \mu) = \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2)R_X(t-\tau_1, \mu-\tau_2) \quad (1.31)$$

If $R_X(t-\tau_1, \mu-\tau_2) = R_X(t-\mu-\tau_1+\tau_2)$ (stationary)

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2 \quad (1.32)$$

Stationary input, Stationary output

$$R_Y(0) = E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2 \quad (1.33)$$
Define: Power Spectral Density (Fourier Transform of $R(\tau)$)

\[ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-2\pi f \tau) d\tau \]  

(1.38)

\[ E[Y^2(t)] = \int_{-\infty}^{\infty} \left| H(f) \right|^2 S_X(f) df \]  

(1.39)

Recall \[ E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \]  

(1.33)

Let $|H(f)|$ be the magnitude response of an ideal narrowband filter

\[ |H(f)| = \begin{cases} 
1, & |f \pm f_c| < \frac{1}{2} \Delta f \\
0, & |f \pm f_c| > \frac{1}{2} \Delta f 
\end{cases} \]  

(1.40)

$\Delta f$: Filter Bandwidth

If $\Delta f \ll f_c$ and $S_X(f)$ is continuous,

\[ E[Y^2(t)] \approx 2\Delta f S_X(f_c) \] in W/Hz
Properties of The PSD

\[ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) \, d\tau \]  \hspace{1cm} (1.42)

\[ R_X(\tau) = \int_{-\infty}^{\infty} S_X(\tau) \exp(j2\pi f \tau) \, df \]  \hspace{1cm} (1.43)

Einstein-Wiener-Khintchine relations:

\[ S_X(f) \leftrightarrow R_X(\tau) \]

\[ S_X(f) \] is more useful than \( R_X(\tau) \)!
Example 1.5 Sinusoidal Wave with Random Phase

\( X(t) = A \cos(2\pi f_c t + \Theta) \),  \( \Theta \sim U(-\pi, -\pi) \)

\( R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau) \)

\( S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau) \, d\tau \)

\( = \frac{A^2}{4} \int_{-\infty}^{\infty} [\exp(j2\pi f_c \tau) \, d\tau + \exp(-j2\pi f_c \tau)] \exp(-j2\pi f \tau) \, d\tau \)

\( = \frac{A^2}{4} [\delta(f - f_c) + \delta(f + f_c)] \)

⇒ Appendix 2, \( \int_{-\infty}^{\infty} \exp[j2\pi (f_c - f)] \, d\tau = \delta(f - f_c) \)
Relation Among The PSD of The Input and Output Random Processes

Recall (1.32)

\[ R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2) \, d\tau_1 \, d\tau_2 \] (1.32)

\[ S_Y(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2) \exp(-j2\pi f \tau) \, d\tau_1 \, d\tau_2 \, d\tau \]

Let \( \tau - \tau_1 + \tau_2 = \tau_0 \) , or \( \tau = \tau_0 + \tau_1 - \tau_2 \)

\[ S_Y(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_1) \exp(j2\pi f \tau_0) \exp(-j2\pi f \tau_2) \exp(-j2\pi f \tau_0) \, d\tau_1 \, d\tau_2 \, d\tau_0 \]

\[ = S_X(f) \cdot H(f) \cdot H^*(f) \]

\[ = |H(f)|^2 S_X(f) \] (1.58)
· White noise

\[
S_W(f) = \frac{N_0}{2}
\]

\[N_0 = kT_e\]  \hspace{1cm} (1.93)

\[T_e : \text{equivalent noise temperature of the receiver}\]

\[
R_W(\tau) = \frac{N_0}{2} \delta(\tau)
\]

\[ (1.95) \]
Example 1.10 Ideal Low-Pass Filtered White Noise

\[ S_N(f) = \begin{cases} \frac{N_0}{2} & -B < f < B \\ 0 & |f| > B \end{cases} \]  \hspace{1cm} (1.96)

\[ R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f \tau) \, df \]  \hspace{1cm} (1.97)

\[ = N_0 B \text{sinc}(2B \tau) \]
Example 1.12 Ideal Band-Pass Filtered White Noise

\[ R_N(\tau) = \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau)df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau)df \]
\[ = N_0 B \text{sinc}(2B\tau) \left[ \exp(-j2\pi f_c\tau) \exp(j2\pi f_c\tau) \right] \]
\[ = 2N_0 B \text{sinc}(2B\tau) \cos(2\pi f_c\tau) \quad (1.103) \]

Compare with (1.97) (a factor of \( \tau \)),

\[ R_{N_I}(\tau) = R_{N_Q}(\tau) = 2N_0 B \text{sinc}(2B\tau). \]