Problem 1: Controllability over time intervals. Given a linear time varying system $R(\cdot) = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$, show that if $R(\cdot)$ is completely controllable on $[t_0, t_1]$, then $R(\cdot)$ is completely controllable on any $[t'_0, t'_1]$, where $t'_0 \leq t_0 < t_1 \leq t'_1$. Show that this is no longer true when the interval $[t_0, t_1]$ is not a subset of $[t'_0, t'_1]$.

Problem 2: Observability Tests for LTI Systems.
Consider the following theorem (page 14 of Lecture Notes 17):

Theorem: The LTI system represented by $(A, C)$ is completely observable on some $[0, \Delta]$ (a) $\iff$ rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$ (b) $\iff$ rank $\begin{bmatrix} sI - A \\ C \end{bmatrix} = n, \forall s \in \sigma(A)$ (c)

Prove the following 4 directions: (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (a), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (b).

One way to prove these is to consider the matrices $(A^T, C^T)$ and follow the controllability results directly (why does this work?).

Problem 3: Feedback control design by eigenvalue placement. Consider the dynamic system:

$$\frac{d^4 \theta}{dt^4} + \alpha_1 \frac{d^3 \theta}{dt^3} + \alpha_2 \frac{d^2 \theta}{dt^2} + \alpha_3 \frac{d\theta}{dt} + \alpha_4 \theta = u$$

where $u$ represents an input force, $\alpha_i$ are real scalars. Assuming that $\frac{d\theta}{dt}, \frac{d^2 \theta}{dt^2}$, and $\theta$ can all be measured, design a state feedback control scheme which places the closed-loop eigenvalues at $s_1 = -1, s_2 = -1, s_3 = -1 + j1, s_4 = -1 - j1$.

Problem 4: Observer design.

Figure 1: Simple model of a DC Servo system, for Problem 9.
Figure 1 shows a block diagram representation of a simple model of a DC servo system: $x_1$ is a voltage signal proportional to the output angular velocity $x_2$.

(a) Design a full order observer, with observer gain matrix $T$ given by

$$ T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, $$

for $x_1$ and $x_2$ so that the characteristic polynomial associated with the error dynamics is given by:

$$ \Delta_e(s) = s^2 + 2\zeta \omega_e s + \omega_e^2 $$

(“Design” means write the equations for the observer, with expressions for gains $T_1$ and $T_2$.)

(b) Now, the observer is a system with inputs $u$ and $x_1$, and outputs $\hat{z}_1$ and $\hat{z}_2$. Thus, there are four possible transfer functions between inputs and outputs – these may be included as elements in a $2 \times 2$ matrix. Evaluate the following matrix of transfer functions $M(s)$ between the inputs to the observer $u$ and $x_1$, and its outputs $\hat{z}_1$ and $\hat{z}_2$:

$$ M(s) = \begin{bmatrix} \hat{z}_1(s)/u(s) & \hat{z}_1(s)/x_1(s) \\ \hat{z}_2(s)/u(s) & \hat{z}_2(s)/x_1(s) \end{bmatrix} $$

as a function of gains $T_1$ and $T_2$, as well as system parameters $a_1$ and $a_2$.

(c) Now determine $M(s)$ as $T_2 \to \infty$. Discuss the meaning of the result.

Problem 5: Control of a Flexible Robot Arm.

A simplified model for the control of a flexible robotic arm is shown in Figure 2. Here, $k$ is a spring constant which models the flexibility of the arm, $M$ represents the mass of the arm, $y$, the output, is the mass position, and $u$, the input, is the position of the end of the spring. Here, $k/M = 900 \text{ rad/s}^2$.

![Figure 2: A simple robotic arm.](image)

The equations of motion for this system are thus given by $M\ddot{y} + k(y - u) = 0$. Define state variables $x_1 = y$, $x_2 = \dot{y}$.

(a) Write the equations of motion in state space form. Where are the open loop eigenvalues?

(b) Design a full state observer with observer eigenvalues at $s = -100 \pm 100j$.

(c) Could both state-variables of the system be estimated if only a measurement of $\dot{y}$ were available?

(d) Design a state feedback controller with gain matrix $F$ giving the closed loop system roots at $s = -20 \pm 20j$.

(e) Would it be reasonable to design a control law for the system with roots at $s = -200 \pm 200j$? State why, or why not.