Problem 1: Internal exponential stability implies BIBO stability (6 points).
Consider the LTI system \( \dot{x} = Ax + Bu, \ y = Cx \).
(a) Prove that if this system is internally exponentially stable, then it is BIBO stable.
(b) Does the same result hold if the system is only internally stable? Prove or give a counterexample.

Problem 2: Controllability (5 points).
For an LTI system \( R = (A, B, C, D) \) with \( A \in \mathbb{R}^{n \times n} \), directly prove the standard controllability result that \( \text{rank}[sI - A|B] = n, \forall s \in \mathbb{C} \) implies that \( \text{rank}[B \ AB \cdots A^{n-1}B] = n \).

Problem 3: Characteristic and minimal polynomials (3 points).
Let the characteristic polynomial of \( A \) be \( \chi_A(s) = (s - \lambda)^5 \) and its minimal polynomial be \( \Psi_A(s) = (s - \lambda)^3 \). True or False: there exists a nonsingular matrix \( P \) such that

\[
PAP^{-1} = \begin{bmatrix}
\lambda & 1 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda
\end{bmatrix}
\]

Explain your answer.

Problem 4: Stability (4 points).
You are given a SISO transfer function \( \frac{(s + 1)(s + 3)}{s^2(s + 2)^2(s + 4)} \) and are told that it has a minimal (controllable and observable) realization \( A, b, c \) with \( A \in \mathbb{R}^{5 \times 5}, b \in \mathbb{R}^5, c \in \mathbb{R}^{1 \times 5} \). Is the system without input: \( \dot{x} = Ax \) stable? Is it asymptotically stable? Prove your conclusions.

Problem 5: Stabilizability and Detectability (4 points).
Consider the LTI system \( \dot{x} = Ax + Bu, \ y = Cx \) where
\[
A = \begin{bmatrix}
-1 & 0 \\
0 & 3
\end{bmatrix}, \ B = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \ C = [0 \ 1]
\]
Is the system stabilizable? Detectable? Prove your answers.
Problem 6: Controllability and observability (6 points).

Consider the LTI system $\dot{x} = Ax + Bu$.

(a) (2 points) Suppose that this open loop system is controllable. Show that the closed loop system resulting from state feedback $u = Fx + v$ is controllable (from new input $v$).

(b) (4 points) Now, in addition, assume that $y = Cx$ and that the open loop system is observable. Suppose that $\psi(y)$ is a known, possibly nonlinear, function of $y$. Show, by designing an appropriate observer and analyzing the convergence of the state estimate error, that the system resulting from output injection: $\dot{x} = Ax + \psi(y) + Bu$, $y = Cx$, is observable. (Hint: consider the block diagram of the observer)

Problem 7: Eigenvalue assignment (10 points).

Consider the SISO LTI system:

$$\dot{x} = Ax + Bu, \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}$$

where

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

(a) Is this system controllable? Under what conditions is the system stabilizable?

(b) Determine the characteristic polynomial of the closed loop system for $u = Kx$, where $K = [k_1 \ k_2 \ \cdots \ k_n]$.

(b) Suppose that you are given $n$ complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ as desired locations for the closed loop eigenvalues. Select the $K$ that would result in the desired values for the closed loop eigenvalues.

(c) Suppose that

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find a matrix $K$ for which the closed loop eigenvalues are $-1, -1, -2$.

Problem 8: Reachable states (5 points).

Given the system

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u; \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(3)

True or false: there exists $u[0, T]$ such that $x(T) = [1, 1, 0]^T$.

If true, give a sketch of the proof, if false, explain.

Problem 9. Controllability of LTI systems (10 points).

(a) Consider the controllability and observability grammmians $W_c, W_o$ of a linear time-invariant system $(A, B, C)$ over the time period $[0, \Delta]$. Determine what happens to them under similarity transformations of the state space. That is, determine the controllability and observability grammmians of $(TAT^{-1}, TB, CT^{-1})$. Prove that the eigenvalues of the product $W_c W_o$ are constant under similarity transformations.

(b) Consider a single input system

$$\dot{x} = Ax + bu$$

(4)

with $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. State necessary and sufficient conditions for complete controllability. Now generalize to the multiple input case.