Problem 1. Note that from the read-out map we have, \( y^T_{\tau} Q y_{\tau} = x^T_{\tau} C^T Q C x_{\tau} \). Thus the original LQR problem can be re-written as:
\[
\min_{x,u} \sum_{\tau=0}^{N-1} \left( x^T_{\tau} \bar{Q} x_{\tau} + u^T_{\tau} R u_{\tau} \right),
\]
which is the standard LQR problem with \( \bar{Q} \) as the state penalty matrix and \( Q_f = 0 \). The optimal cost-to-go and the optimal control at time \( t \) are thus given by:
\[
J^*_t(z) = z^T P_t z \tag{1}
\]
\[
u^*_t = -K_t z, \tag{2}
\]
where \( t \in \{0,1,\ldots,N-1\} \) and
\[
P_t = \bar{Q} + K^T_t R K_t + (A - BK_t)^T P_{t+1} (A - BK_t), \quad P_N = 0 \tag{3}
\]
\[
K_t = (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A. \tag{4}
\]

Problem 2. We can code up the optimal control policy derived in Problem-1 to analyze the system. Here are the plots for the optimal control, output and cost-to-go for the three cases:

![Figure 1: Used control authority](image-url)
Since the output penalty is higher in case (ii), output is quickly driven to zero compared to the other two cases (see Figure 2). To do so, a higher control authority is used, as evident from Figure 1. On the other hand, when input penalty is higher, the control becomes very expensive. Thus, used control magnitude is very small as evident from Figure 1. As a result, output is not driven to zero even by the end of the horizon.

**Problem 3.** We assume that this problem is in the linear dynamics case, i.e. by "not dynamically feasible" we can assume the specific case of $Ax_t^* + Bu_t^* \neq x_{t+1}^*$. Define time-varying constant $c_t$ that captures the deviation between the feasible trajectory’s linear dynamics and the actual next state:

$$c_t = (Ax_t^* + Bu_t^*) - x_{t+1}^*$$
\[ x^*_{t+1} = (Ax^*_t + Bu^*_t) - c_t \]

which is an affine equation. Define new state: \( z_t = x_t - x^*_t \), and new control \( v_t = u_t - u^*_t \) therefore:

\[
z_{t+1} = x_{t+1} - x^*_{t+1} = x_{t+1} - ((Ax^*_t + Bu^*_t) - c_t) \\
= Ax_t + Bu_t - Ax^*_t - Bu^*_t + c_t \\
= A(x_t - x^*_t) + B(u_t - u^*_t) + c_t \\
= A(z_t) + B(v_t) + c_t
\]

Now let’s augment our state variable to handle the constant term. Define new state: \( \gamma_t = \begin{bmatrix} z_t \\ 1 \end{bmatrix} \)

\[
\gamma_{t+1} = \begin{bmatrix} z_{t+1} \\ 1 \end{bmatrix} = \tilde{A} \begin{bmatrix} z_t \\ 1 \end{bmatrix} + \tilde{B}v_t
\]

where \( \tilde{A} = \begin{bmatrix} A & c_t \\ 0 & 1 \end{bmatrix} \), \( \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \). Because we augmented the state matrices we pad the \( Q \) and \( Q_f \) matrices accordingly.

When dealing with non-dynamically feasible references (i.e. this problem), we get a time-varying affine system, and not a linear system. The set of LQR equations look slightly different for the affine systems (they are still the same for TV and TI systems, but they are different than that for linear systems).

We can see the difference if we re-derive the equations. The \( P \) and \( K \) matrices are the same as the time-invariant case, but the control law should be:

\[
u_t = u^*_t - K_t \begin{bmatrix} z_t \\ 1 \end{bmatrix}\]