Problem 1. Choose \( r, \theta, \dot{r} \) and \( \dot{\theta} \) as the states of the system (note that an argument could be made for selecting just \( \dot{r} \) and \( \dot{\theta} \)). The Jacobian of 

\[
\begin{bmatrix}
\dot{r} \\
\dot{\theta} \\
r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\
-2\frac{\dot{r}}{r} + \frac{1}{r}u_2
\end{bmatrix}
\]

w.r.t. the states is given by

\[
D_1 f(r, \theta, \dot{r}, \dot{\theta}, u) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
3\omega^2 & 0 & 0 & 2p\omega \\
0 & 0 & -\frac{2\omega}{p} & 0
\end{bmatrix}
\]

On the reference trajectory we have \( u^{\text{ref}}(t) \equiv 0, r^{\text{ref}}(t) \equiv p, \dot{r}^{\text{ref}}(t) \equiv 0, \theta^{\text{ref}}(t) = \omega t, \) and \( \dot{\theta}^{\text{ref}}(t) = \omega \). Note that the only time-varying part of the reference trajectory is \( \theta^{\text{ref}}(t) = \omega t \), but the Jacobians happen to not have \( \theta \), so this is a case where linearizing about this trajectory yields time-invariant matrices \( A \) and \( B \) (in general you’d get time-varying \( A \) and \( B \)).

Evaluating the Jacobian at the reference trajectory gives

\[
D_1 f(r, \theta, \dot{r}, \dot{\theta}, u)|_{\text{at ref}} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
3\omega^2 & 0 & 0 & 2p\omega \\
0 & 0 & -\frac{2\omega}{p} & 0
\end{bmatrix}
\]

Similarly, the Jacobian of \( f \) w.r.t. the input \( u \) is

\[
D_2 f(\dot{r}, \dot{\theta}, u) = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & \frac{1}{r}
\end{bmatrix}
\]

and so

\[
D_2 f(\dot{r}, \dot{\theta}, u)|_{\text{at ref}} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & \frac{1}{p}
\end{bmatrix}
\]

Therefore the linearized equation about the reference orbit is

\[
\begin{bmatrix}
\frac{d}{dt}\delta r \\
\frac{d}{dt}\delta \theta \\
\frac{d}{dt}\delta \dot{r} \\
\frac{d}{dt}\delta \dot{\theta}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
3\omega^2 & 0 & 0 & 2p\omega \\
0 & 0 & -\frac{2\omega}{p} & 0
\end{bmatrix} \begin{bmatrix}
\delta r \\
\delta \theta \\
\delta \dot{r} \\
\delta \dot{\theta}
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & \frac{1}{p}
\end{bmatrix} \begin{bmatrix}
\delta u_1 \\
\delta u_2
\end{bmatrix}
\]
Problem 2.

(a) From our knowledge of linear systems, we know that the state transition matrix obeys the differential equation \( \dot{\Phi}(t, t_0) = A(t) \Phi(t, t_0) \) and initial condition \( \Phi(t_0, t_0) = I \). Because this is a time-invariant system, without loss of generality, let \( t_0 = 0 \). If we denote the \( i,j \)th entry of \( \Phi(t, t_0) \) as \( \phi_{i,j} \), then we can write down the following four differential equations

\[
\begin{align*}
\dot{\phi}_{1,1} &= -\phi_{1,1}, & \phi_{1,1}(0) &= 1 \\
\dot{\phi}_{1,2} &= -\phi_{1,2}, & \phi_{1,2}(0) &= 0 \\
\dot{\phi}_{2,1} &= 2\phi_{1,1} - 3\phi_{2,1}, & \phi_{2,1}(0) &= 0 \\
\dot{\phi}_{2,2} &= 2\phi_{1,2} - 3\phi_{2,2}, & \phi_{2,2}(0) &= 1 
\end{align*}
\]

We can solve directly for \( \phi_{1,1} \) and \( \phi_{1,2} \), and we see that \( \phi_{1,1}(t) = e^{-t} \) and \( \phi_{1,2}(t) = 0 \). With these solutions, we can then solve directly for \( \phi_{2,2} \), obtaining \( \phi_{2,2}(t) = e^{-3t} \). We can solve for \( \phi_{2,1} \) by finding an integrative factor \( \mu(t) \) such that \( \dot{\mu}(t) = 3\mu(t) \) (this implies that \( \mu = e^{3t} \)).

\[
\begin{align*}
\dot{\phi}_{2,1} + 3\phi_{2,1} &= 2e^{-t} \\
\mu\phi_{2,1} + 3\mu\phi_{2,1} &= \mu2e^{-t} & (\text{multiply by } \mu) \\
\frac{d}{dt}\mu\phi_{2,1} &= \mu2e^{-t} & (\text{product rule for } \mu\phi_{2,1}) \\
\mu\phi_{2,1} + c &= \int \mu2e^{-t} & (\text{integrate both sides}) \\
ke^{3t}\phi_{2,1} + c &= \int ke^{3t}2e^{-t} & (\text{substitute for } \mu) \\
ke^{3t}\phi_{2,1} &= ke^{2t} + c & (\text{solve and rearrange constants}) \\
\phi_{2,1} &= e^{-t} + ce^{-3t} & (\text{multiply by } k^{-1}e^{-3t}) \\
\phi_{2,1} &= e^{-t} - e^{-3t} & (\text{solve for } c \text{ using initial conditions})
\end{align*}
\]

Therefore, our solution \( \Phi(t, t_0) \) is given by

\[
\Phi(t, 0) = \begin{bmatrix}
e^{-t} & 0 \\
e^{-t} - e^{-3t} & e^{-3t}
\end{bmatrix}
\]

Note that because \( A \) is not time-varying, \( \phi(t, t_0) = e^{At} \), so we could have computed the matrix exponential instead (generally faster method when have linear time-invariant system).

(b) We can follow the same procedure as outlined in part (a). However, because this is a time-varying system, we must consider the general case of \( t_0 \neq 0 \). Note that, in this case, we obtain a difficult to evaluate integral when we solve for \( \phi_{2,1} \). It is okay to leave the expression for \( \phi_{2,1} \) in integral form (this problem illustrates how even a simple time dependence in the \( A \) matrix can make computing the state transition matrix difficult). Because the state transition matrix is defined over the time interval \( t_0 \) to
t, technically the integral should be a definite integral from $t_0$ to $t$:
\[
\Phi(t, t_0) = \begin{bmatrix} e^{-(t^2-t_0^2)} & 0 \\
\int_{t_0}^{t} e^{-(\tau^2-t_0^2)} - \phi_{12}(\tau) d\tau + c & e^{-(t-t_0)} \end{bmatrix}
\]

If you do compute the integrating factor, it would be $e^t$ in this case, and you could simplify the integral to:
\[
\Phi(t, t_0) = \begin{bmatrix} e^{-(t^2-t_0^2)} & 0 \\
e^{-t} \int_{t_0}^{t} e^{-(\tau^2-t_0^2)} d\tau + ce^{-t} & e^{-(t-t_0)} \end{bmatrix}
\]

(c) Remember that because this is a time-varying system, we cannot assume $t_0 = 0$. Taking inspiration from the hint provided, we define $\tilde{\Omega}(t, t_0) = \int_{t_0}^{t} \omega(\tau) d\tau$ (or equivalently $\tilde{\Omega}(t, t_0) = \int_{t_0}^{t} \omega(\sigma + t_0) d\sigma$, with change of variables $\tau = \sigma - t_0$). We note that $\tilde{\Omega}(t_0, t_0) = 0$, and using Leibniz’ Rule, we find that $\frac{d}{d\tau}\tilde{\Omega}(t, t_0) = \omega(t)$. We further define the candidate matrix $H(t, t_0)$ using a similar structure to the matrix hint:
\[
H(t, t_0) = \begin{bmatrix} \cos \tilde{\Omega}(t, t_0) & \sin \tilde{\Omega}(t, t_0) \\
-\sin \tilde{\Omega}(t, t_0) & \cos \tilde{\Omega}(t, t_0) \end{bmatrix}
\]

We see that the derivative of this matrix with respect to time is
\[
\dot{H}(t, t_0) = \begin{bmatrix} -\omega(t) \sin \tilde{\Omega}(t, t_0) & \omega(t) \cos \tilde{\Omega}(t, t_0) \\
-\omega(t) \cos \tilde{\Omega}(t, t_0) & -\omega(t) \sin \tilde{\Omega}(t, t_0) \end{bmatrix}
\]
which is equal to $A(t)H(t, t_0)$. Furthermore, at $t_0$, we see that $H(t_0, t_0) = I$. Therefore, we can see that $H(t, t_0)$ satisfies the differential equation and initial conditions of the state transition matrix and is therefore our (unique) solution.
\[
\Phi(t, t_0) = H(t, t_0)
\]

(d) The zero input (non-zero initial state) response is described by
\[
\rho(t, t_0, x_0, 0) = r(t, \Phi(t, t_0)x_0, 0)
\]

For system (a), we see that all the terms in the state transition matrix $\Phi(\cdot, \cdot)$ are exponentials raised to a negative power. This means that each term in the matrix exponential decays to zero as $t \to \infty$, i.e. the response tends to zero as well for any $x_0 \neq 0$.

For system (b), we have a similar behavior where each term decays to zero as $t \to 0$. Without evaluating the ugly integral for $\phi_{21}(t)$, notice that from the differential equation, $x_1 \to 0$, resulting in $\dot{x}_2 = -x_2$, which causes $x_2 \to 0$.

**Problem 3.** We show the matrix exponential is invertible using properties 1 and 3 from 221A lecture 10 page 4. First, we want to show $\exists$ an $M$ such that $Me^{At} = I$. We propose a candidate expression for $M$ and show it satisfies this equation. Try $M = e^{-At}$, then
\[
e^{-At} e^{At} = e^{(-A+A)t} \quad \text{by property 3}
\]
\[
e^{0t} = I \quad \text{by property 1}
\]
Thus $e^{At}$ is invertible (full rank and bijective too!), and the inverse is $e^{-At}$.

**Problem 4.** $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

First, $[sI - A]^{-1} = \begin{bmatrix} s & -1 & 0 \\ -1 & s & 1 \\ 0 & 0 & s \end{bmatrix}^{-1} = \frac{1}{\det(sI - A)} \cdot \text{adjugate}(sI - A)$ \hspace{1cm} (3)

$= \frac{1}{s(s^2 - 1)} \begin{bmatrix} s^2 & s & -1 \\ s & s^2 & -s \\ 0 & 0 & s^2 - 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2 - 1} & \frac{1}{s} & \frac{-1}{s^2 - 1} \\ \frac{1}{s^2 - 1} & \frac{s}{s^2 - 1} & \frac{-1}{s} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$ \hspace{1cm} (4)

Some Laplace transform tables don’t have many listed pairs so these elements may not match in the table. In that case, we use partial fraction expansion. This gives

$[sI - A]^{-1} = \begin{bmatrix} \frac{1}{s + 1} + \frac{1}{s - 1} & -\frac{1}{s + 1} + \frac{1}{s - 1} & \frac{1}{s} - \frac{1}{s + 1} - \frac{1}{s - 1} \\ \text{same as (1,2) entry} & \text{same as (1,1) entry} \end{bmatrix}$ \hspace{1cm} (5)

Then taking the inverse Laplace transform we get

$\mathcal{L}^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^t & -\frac{1}{2}e^{-t} + \frac{1}{2}e^t & 1 - \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ \text{same as (1,2) entry} & \text{same as (1,1) entry} & \frac{1}{2}e^{-t} - \frac{1}{2}e^t \\ 0 & 0 & 1 \end{bmatrix}$ \hspace{1cm} (6)

These matrix elements also fit Laplace transform tables that include the cosh and sinh functions. Using these, we can equivalently express the matrix exponential as

$\begin{bmatrix} \cosh(t) & \sinh(t) & 1 - \cosh(t) \\ \sinh(t) & \cosh(t) & -\sinh(t) \\ 0 & 0 & 1 \end{bmatrix}$ \hspace{1cm} (7)

**Problem 5.** For reference, this problem is asking you to show the "exact discretization" process, where we express the discrete time solution in terms of the continuous time matrices. This process is called exact in contrast to an approximation of the continuous time dynamics at discrete steps, such as what’s done in Euler’s method.

**(a)** While this is a sampled data system, the dynamics are still fundamentally continuous. Therefore, we begin by writing the general form for a solution to a continuous linear time-invariant system

$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\eta)}Bu(\eta)d\eta$ \hspace{1cm} (8)

We seek a formula for $x((k+1)T)$ in terms of $x(kT)$ and $u(t)$. We can use $t_0 = kT$ and $x_0 = x(kT)$ in the above equation, and we can evaluate the expression at $t = (k+1)T$.

$x((k+1)T) = e^{A((k+1)T-kT)}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\eta)}Bu(\eta)d\eta$
We note that, over the time interval of integration, \( u(t) \) is constant and equal to \( u(k) \), so we can pull the term \( Bu(\eta) \) out of the integral to the right. Furthermore, we can rewrite the integral using the substitution \( \tau = \eta - kT \). Lastly, we can use the notation \( x(kT) = x(k) \) and \( x((k+1)T) = x(k+1) \). The equation becomes

\[
x(k + 1) = e^{AT}x(k) + \left( \int_0^T e^{A(T-\tau)}d\tau \right) Bu(k)
\]

which provides our formula for \( x(k + 1) \) in terms of \( x(k) \) and \( u(k) \).

(b) Suppose that, in addition, the readout map of this discrete-time linear system is given by \( y(k) = Cx(k) + Du(k) \). The response function of the discrete-time linear system, in terms of \( k \), \( 0 \), \( x(0) \) and the input sequence \( \{u(0),...,u(k)\} \) is:

\[
\rho(k, 0, x(0), \{u(0),...,u(k)\}) = Cx(k) + Du(k)
\]

\[
= C\left( e^{AT}x(k) + \left( \int_0^T e^{A(T-\tau)}d\tau \right) Bu(k) \right) + Du(k)
\]

Above is from backwards applying the discrete time update equation. Then we keep doing this until we get an expression in terms of \( k \), \( 0 \), \( x(0) \) and the input sequence \( \{u(0),...,u(k)\} \).

\[
= C\left( e^{kAT}x(0) + \left( \int_0^T e^{A(T-\tau)}d\tau \right) B\left( e^{(k-1)AT}u(0) + \ldots + u(k-1) \right) \right) + Du(k)
\]

Letting \( \bar{A} = e^{AT} \) and \( \bar{B} = \left( \int_0^T e^{A(T-\tau)}d\tau \right) B \), we can write the final response function as:

\[
\rho(k, 0, x(0), \{u(0),...,u(k)\}) = C\left[ \bar{A}^kx(0) + \bar{B} \sum_{i=0}^{k-1} \bar{A}^{k-1-i}u(i) \right] + Du(k)
\]