Problem 1. (a) **Way 1:** Consider \( A = U \Sigma V^* \), then \( A^* A = V \Sigma^2 V^* \). Let \( v_i \) be the \( i \)th column of \( V \), then \( v_i^* \) is the \( i \)th row of \( V^* \). Now let’s define what the decomposition of \( A^* A \) is

\[
A^* A = \sum_{i=1}^{n} \sigma_i^2 v_i v_i^*
\]

Next we want to get \( x_0 \) in a similar form. We know \( \{v_1, v_2, \ldots, v_n\} \) forms a basis for \( \mathbb{R}^n \) (full rank, spans entire space, orthogonal vectors). Therefore, we can represent any \( x_0 \in \mathbb{R}^n \) as:

\[
x_0 = \sum_{j=1}^{n} \alpha_j v_j
\]

Putting together the two:

\[
x_1 = A^* A x_0
\]

\[
= (\sum_{i=1}^{n} \sigma_i^2 v_i v_i^*) (\sum_{j=1}^{n} \alpha_j v_j)
\]

\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} (\sigma_i^2 v_i v_i^*) (\alpha_j v_j)
\]

Because \( V \) is unitary, \( v_i^* v_j = \{1 \text{ if } i = j, 0 \text{ otherwise} \} \). Therefore we can write:

\[
x_1 = \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_i^2 \alpha_j v_i = \sum_{i=1}^{n} \sigma_i^2 \alpha_i v_i
\]

\[
x_2 = A^* Ax_1 = (\sum_{i=1}^{n} \sigma_i^2 v_i v_i^*) (\sum_{j=1}^{n} \sigma_i^2 \alpha_i v_i) = \sum_{i=1}^{n} \sigma_i^4 \alpha_i v_i
\]

\[
\vdots
\]

\[
x_k = A^* Ax_{k-1} = \sum_{i=1}^{n} \sigma_i^{2k} \alpha_i v_i
\]

Now assume \( \sigma_1 > \sigma_2 > \cdots > \sigma_n \).

\[
\frac{||x_1||^2}{||x_0||^2} \approx \frac{\sigma_1^4 \alpha_1^2}{\alpha_1^2} \approx \sigma_1^4
\]

\[
\frac{||x_2||^2}{||x_1||^2} \approx \frac{\sigma_1^8 \alpha_1^2}{\sigma_1^4 \alpha_1^2} \approx \sigma_1^4
\]

**Way 2:**

Generalize the \( k \)'th element of this sequence in terms of \( x_0 \) by substituting \( x_1, x_2, \ldots, x_k \) down to \( x_0 \), yielding the \( k \)'th element of the sequence \( \frac{||x_{k+1}||^2}{||x_k||^2} = \frac{||(A^* A)^{k+1} x_0||^2}{||(A^* A)^k x_0||^2} \).

Then substituting the SVD for \( A \) and simplifying, we have \( ||x_k||^2 = x_0^* V \Sigma^2 \Sigma^2 \Sigma^2 \Sigma^2 V^* x_0 \).
Thus \( \frac{\|x_{k+1}\|^2}{\|x_k\|^2} = \frac{x_0^\ast \Sigma^{2,k+1} \Sigma^{2,k+1} V^\ast x_0}{x_0^\ast \Sigma^{2,k} \Sigma^{2,k} V^\ast x_0} \).

Using the same change of variables as from the discussion on SVD, let \( \alpha = V^\ast x_0 \).

Then \( \frac{\|x_{k+1}\|^2}{\|x_k\|^2} = \frac{\alpha_1^2 \sigma_1^{2(k+1)} + \ldots + \alpha_r^2 \sigma_r^{2(k+1)}}{\alpha_1^2 \sigma_1^{2k} + \ldots + \alpha_r^2 \sigma_r^{2k}} = \frac{\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^4 + \ldots + \alpha_r^2 \sigma_r^4}{\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^4 + \ldots + \alpha_r^2 \sigma_r^4}. \) Finally, we see that as \( k \to \infty \), all terms except the first of both the numerator and denominator zero out because \( \sigma \) is greater than \( \sigma_2 \ldots \sigma_r \). Thus we are left with convergence to \( \alpha_1^2 \).

(b) **Way 1:** Recall that \( x_0 = \sum_{j=1}^n \alpha_j v_j \). The system will not converge if \( \alpha_1 = 0 \), i.e. the weight of \( x_0 \) along the first eigenvector of \( A^\ast A \) (which is associated with \( \sigma_1 \)). **Way 2:** because \( \alpha = V^\ast x_0 \), then \( x_0 = V \alpha = \sum_{j=1}^n \alpha_j v_j \). So we have the same answer as Way 1.

**Problem 2.** The Jacobian of \( f \) is

\[
D_x f = \begin{bmatrix}
0 & 1 \\
-g/l \cos(x_1) & -k/m
\end{bmatrix}
\]

We can see that \( ||Df||i \) is uniformly bounded on \( \mathbb{R}^2 \). Consider the induced infinity-norm of the Jacobian:

\[
||D_x f||i, \infty = \text{max} \left\{ 1, | - \frac{g}{l} \cos(x_1)| + | - \frac{k}{m} | \right\}
\]

\[
\leq 1 + | - \frac{g}{l} \cos(x_1)| + | - \frac{k}{m} |
\]

\[
\leq 1 + \frac{g}{l} + \frac{k}{m}
\]

where the last inequality holds since \( \forall x_1 \in \mathbb{R}, -1 \leq \cos(x_1) \leq 1 \) and \( x_2 \) is not present in the Jacobian. Since we have found a bound on \( ||D_x f||i \), that holds for all \( x \in \mathbb{R}^2 \), then the system is globally Lipschitz in \( x \).

**Problem 3.** (a) The Jacobian of the function is:

\[
D_x f = 2x
\]

It is locally Lipschitz continuous since \( D_x f = 2x \) is bounded in an arbitrary compact interval. But it is not globally Lipschitz continuous. Assume that it is and we will prove by contradiction. Let \( L \) be the global Lipschitz constant for this function. Then \( ||f(x) - f(y)|| \leq L||x - y||, \forall x, y \in \mathbb{R} \). For any \( L > 0 \), pick \( x = 2L \) and \( y = 0 \). Then \( ||f(x) - f(y)|| = 4L^2 \) and \( ||x - y|| = 2L \). Thus, we see that \( ||f(x) - f(y)|| = 4L^2 \neq 2L^2 = 2L \). Therefore, \( f \) is not globally Lipschitz continuous.

(b) We have a first order nonlinear ODE that is separable, so we solve:

\[
\int \frac{1}{x^2} \dot{x} dt = \int 1 dt
\]

to get \( x(t) = -\frac{1}{t + c} \). After plugging in the initial condition and solving for \( c_1 \), we get the solution to this differential equation to be: \( x(t) = \frac{c}{1 + c(t_0 - t)} \). When \( c = 0 \), the solution is \( x \equiv 0 \), and it is globally defined. If \( c \neq 0 \), then it is well-defined only for \( t \in (-\infty, t_0 + 1/c) \).

**Problem 4.**
(a) Call the first system $\dot{x} = f(x,t)$ and the second one $\dot{x} = g(x)$. Then the Jacobian of $f$ is given by

$$D_x f(x,t) = \begin{bmatrix} -1 - e^t \sin(x_1 - x_2) & +e^t \sin(x_1 - x_2) \\ 15 \cos(x_1 - x_2) & -1 - 15 \cos(x_1 - x_2) \end{bmatrix}$$

and the Jacobian of $g$ is given by:

$$D_x g(x) = \begin{bmatrix} -1 + x_2 & x_1 \\ 0 & -1 \end{bmatrix}$$

**Lipschitz Continuity of $f(x,t)$:**

$$\|D_x f(x,t)\|_{\infty} \text{ (or } \|D_x f(x,t)\|_{i,1} \text{) is uniformly bounded in } x \text{ since } \sin(x) \text{ and } \cos(x) \text{ are bounded below by } -1 \text{ and above by } 1, \forall x \in \mathbb{R}. \text{ So:}$$

$$\|D_x f(x,t)\|_{\infty} = \max \left\{ | -1 - e^t \sin(x_1 - x_2)| + |e^t \sin(x_1 - x_2)|, |15 \cos(x_1 - x_2)| + | -1 - 15 \cos(x_1 - x_2)| \right\}$$

$$\leq | -1 - e^t \sin(x_1 - x_2)| + |e^t \sin(x_1 - x_2)| + |15 \cos(x_1 - x_2)| + | -1 - 15 \cos(x_1 - x_2)|$$

$$\leq | -1 - e^t| + e^t + 15 + | -1 - 15|$$

$$\leq 31 + 2e^t$$

This final RHS is indeed upper bounded by a lipschitz constant $k(t)$, which we defined in lecture 7 as simply a piecewise continuous function in time $t$. Even though $e^t$ is unbounded it is still piecewise continuous. Thus, we conclude that $f(x,t)$ is globally Lipschitz continuous in $x$.

**Lipschitz Continuity of $g(x)$:**

For the second system, from looking at the nonlinear term $x_1 x_2$ and knowing some examples of not globally lipschitz functions, we have a suspicion that it is not globally lipschitz. Thus let’s try to prove not globally lipschitz in a proof by contradiction. Assume that it is globally Lipschitz continuous, and then there exists a constant $L$ such that $\|g(x) - g(y)\| \leq L \|x - y\|$. Let $x = (x_1, x_2) = (0,0)$ and set $y = (y_1, y_2) = (2L, -2L)$ for some $L > 0$, then $\|g(x) - g(y)\|_{\infty} = \|(2L + 4L^2, -2L)\|_{\infty} \leq 4L^2 + 4L \not\leq 2L^2 = L \|(2L, -2L)\|_{\infty} = L \|x - y\|_{\infty}$. Therefore, $g$ is not globally Lipschitz continuous, and you can find it is locally lipschitz from taking the norm of the jacobian.

(b) You should agree. Note that $x_2$ does not depend on $x_1$; it satisfies the conditions of the fundamental theorem (piecewise continuous in $t$ and Lipschitz continuous in $x$), and one can directly find the unique solution for $\dot{x}_2 = -x_2$ which is $x_2(t) = x_2(0)e^{-t}$. Additionally, $x_2(t) \to 0$ as $t \to \infty$. This solution for $x_2$ can be substituted into the first equation, and because the system is time-invariant ($f$ does not depend on $t$) we can assume $t_0 = 0$. This gives us $\dot{x}_1 = x_1(x_2(0)e^{-t} - 1)$, which again satisfies the conditions of the fundamental theorem.

We can obtain the unique solution $x_1(t)$ of this homogeneous equation: before incorporating the initial condition we have $x_1(t) = e^{(x_2(0)e^{-t} - t)+c}$, where $c$ is the integration constant. Then from evaluating at $t = 0$ we get $e^c = x_1(0)e^{x_2(0)}$. Thus the final solution in terms of initial conditions $x_1(0)$ and $x_2(0)$ is $x_1(t) = x_1(0)e^{(x_2(0)(1-e^{-t})-t)}$.

This solution $x_1(t)$ also tends to zero as $t \to \infty$, for any $x_1(0)$ and $x_2(0)$. Thus, even though the diff eq isn’t globally lipschitz continuous, by solving the system we see we have a unique solution for each
initial condition. For other locally lipschitz systems we may not have had this unique solution property, so we need to solve the system of equations to tell.