EE221A Section 6

October 2, 2020

Based on Lectures 6 and 7

Topics: $E$ and $U$ Theorem of Diff Eqs, BG Lemma, Dynamical Systems, Intro to LTI systems

1 Fundamental Theorem

Theorem 1 (Fundamental Theorem of Differential Equations). Consider the following ordinary differential equation (ODE):

\[
\dot{x} = f(x, t),
\]

with the vector field \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \). If \( f \) is

- piecewise continuous in \( t \)
- Lipschitz continuous in \( x \),

then the ODE admits a unique solution, which is differentiable almost everywhere except at points where \( f \) is discontinuous with respect to \( t \).

2 Bellman-Gronwall lemma

Theorem 2 (Bellman-Gronwall Inequality). Let \( u(\cdot) \) be a nonnegative, piecewise continuous function on \([0, T]\). If

\[
u(t) \leq c_1 + \int_{t_0}^{t} k(\tau)u(\tau)d\tau
\]

for some constant \( c_1 \geq 0 \) and a nonnegative integrable function \( k \), then

\[
u(t) \leq c_1 \exp \left( \int_{t_0}^{t} k(\tau)d\tau \right),
\]

for \( 0 \leq t_0 < t \leq T \).
Problem 1. (Variation on linear systems) Consider the following linear system:

\[ \dot{x} = Ax(t), \quad t \in (0, T] \]
\[ x(0) = x_0. \]

where the matrix \( A \) is in \( \mathbb{R}^{n \times n} \). Now we consider the variation \( x_0 + \tilde{x}_0 \) of the initial value, and the corresponding linear system:

\[ \dot{\tilde{x}} = A\tilde{x}(t), \quad t \in (0, T] \]
\[ \tilde{x}(0) = x_0 + \tilde{x}_0. \]

Let \( \tilde{x} := \dot{x} - x \) be the variation on the state. Then \( \tilde{x} \) solves the following linear system:

\[ \dot{\tilde{x}} = A\tilde{x}(t), \quad t \in (0, T] \]
\[ \tilde{x}(0) = \tilde{x}_0. \]

Show that \( \| \tilde{x}(t) \| \to 0 \) as \( \| \tilde{x}_0 \| \to 0 \) for any \( t \in [0, T] \) using Bellman-Gronwall lemma.

Exercise. Derive the more general Bellman-Gronwall lemma, i.e. \( C_1 \) is not a constant but rather a set piecewise function in \( t \), \( C_1(t) \).

Problem 2. (Differential version of B.G. lemma) Let \( x(t) \) be a nonnegative, continuously differentiable function on \([0, T]\), which satisfies

\[ \dot{x}(t) \leq a(t)x(t) + b(t)u(t) \]

for all \( t \in [0, T] \), where \( a, b, \) and \( u \) are nonnegative integrable functions on \([0, T]\). Show that the solution \( x(t) \) is upper bounded by a growing exponential.
3 Dynamical Systems

\((U, Y, \Sigma, s, r)\): (input, state, output, state transition function, output read-out map).

- **Input**: \(U \subset \{ u : [0, \infty) \rightarrow U \mid U \) vector space (typically \(\mathbb{R}^n)\} \) (Note that \(U\) is a function space.)

- **Output**: \(Y \subset \{ y : [0, \infty) \rightarrow Y \mid Y \) vector space (typically \(\mathbb{R}^n)\} \) (Note that \(Y\) is a function space.)

- **State Space**: \(\Sigma\), a vector space (typically \(\mathbb{R}^n)\)

- **State transition function**: \(s : \mathbb{R} \times \mathbb{R} \times \Sigma \times U \rightarrow \Sigma \) with \(s(t, t_0, x_0, u) = x(t)\)

- **Output read-out map**: \(r : \mathbb{R} \times \Sigma \times U \rightarrow Y \) with \(r(t, x(t), u(t)) = y(t)\)

- **Response function**: composition of \(s\) and \(r\): \(\rho : \mathbb{R} \times \mathbb{R} \times \Sigma \times U \rightarrow Y \) with \(\rho(t, t_0, x_0, u[t_0, t]) = y(t)\)

**Consider.** Why are we defining these vector spaces?

**Problem 3.** Suppose that the dynamical system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t) \\
x(t_0) &= x_0 
\end{align*}
\]  

admits the unique solution

\[x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau,\]

for \(t \in [t_0, \infty)\). Identify the state transition function, the output read-out map, the response function, zero input-response (natural), and zero-state response (forced).

**Two axioms**

1. **State transition axiom**: given \(u_1, u_2 \in U\) with \(u_1(t) = u_2(t)\) for \(t \in [t_1, t_2]\), we have

\[
s(t_2, t_1, x_0, u_1[t_1, t_2]) = s(t_2, t_1, x_0, u_2[t_1, t_2])
\]

2. **Semi-group axiom**: \(\forall t_0 \leq t_1 \leq t_2, \forall x_0 \in \Sigma, \forall u \in U,\)

\[
s(t_2, t_0, x_0, u[t_0, t_2]) = s(t_2, t_1, s(t_1, t_0, x_0, u[t_0, t_1]), u[t_1, t_2])
\]
4 Linear and Time-Invariant Dynamical Systems

Definition 3. \((U, Y, \Sigma, s, r)\) is said to be a \textit{linear dynamical system} if

- \(U, \Sigma, Y\) are vector spaces over the same field;
- the response map \(\rho\) is linear in both \(x_0\) and \(u_0\), i.e.,
  \[
  \rho(t_1, t_0, \alpha_1 x_0 + \alpha_2 x_0, \alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \rho(t_1, t_0, x_0, u_1) + \alpha_2 \rho(t_1, t_0, x_0, u_2).
  \]

Definition 4. \((U, Y, \Sigma, s, r)\) is said to be a \textit{time-invariant dynamical system} if

- We define a shift operator \(T_\tau : F \rightarrow F\) for \(F = U\) or \(Y\), such that
  \[
  (T_\tau(f))(t) = f(t - \tau)
  \]
- \(U\) and \(Y\) are closed under \(T_\tau\) for all \(\tau\).
- For all \(t_0, t_1 \geq t_0\), \(\tau \in T\), for all \(x_0 \in \Sigma\), for all \(u \in U\)
  \[
  \rho(t_1, t_0, x_0, u) = \rho(t_1 + \tau, t_0 + \tau, x_0, T_\tau(u)).
  \]

Consider. (Roller coaster) Consider a system that captures the motion of an empty roller coaster on a track at an amusement park. Let the dynamical system states be the coaster’s position and velocity, and let \(t_0\) be the time that the roller coaster is launched. Is this system time-invariant?

Continuous time-varying linear system

\[
\begin{align*}
  \dot{x} &= A(t)x + B(t)u \\
  y &= C(t)x + D(t)u \\
  x(t_0) &= x_0,
\end{align*}
\]

where \(x(t) \in \mathbb{R}, u(t) \in \mathbb{R}, y(t) \in \mathbb{R}\), and \(A, B, C, D\) are matrices whose elements are functions in \(t\).

Problem 4. Is the following system linear? \(x(t) \in \mathbb{R}^2, u(t) \in \mathbb{R}, y(t) = x(t)\) and \(a, b, c, t \in \mathbb{R}\)

\[
\begin{align*}
  \dot{x}_1 &= \ln(t)x + e^{at}u \\
  \dot{x}_2 &= ax_1 x_2 + b^2 u - cu \\
  x(t_0) &= x_0,
\end{align*}
\]
Discrete linear time-invariant system

Note: discrete time systems is not a focus in this class, but this is useful to know more broadly

\[
\begin{align*}
    x_{k+1} &= ax_k + bu_k \\
    y_k &= cx_k + du_k \\
    x(k_0) &= x_0,
\end{align*}
\]

(4)

where \(x_k \in \mathbb{R}, u_k \in \mathbb{R}, y_k \in \mathbb{R},\) and \(a, b, c, d \in \mathbb{R}\.\

**Exercise.** Derive that the solution to the differential equation in \(x\) given by (4) is

\[
x_k = a^k x_0 + \sum_{i=k_0}^{k-1} a^{k-i-1} bu_i
\]

(5)

**Problem 5.** Show that the system in (4) is a) linear and b) time-invariant