1 Singular Value Decomposition

Consider: Why is SVD useful?

Definition 1. A matrix \( M \in \mathbb{R}^{n \times n} \) is called orthogonal if all rows and columns of the matrix are mutually orthogonal. Moreover, if all rows and columns have a unit norm, the matrix is called orthonormal. So for an orthonormal matrix, we have \( M^T M = M M^T = I \), where \( I \) is an identity matrix of size \( n \times n \).

Theorem 2. Any \( m \times n \) matrix can be factored into \( A = U \Sigma V^\top \), where \( U \) is an \( m \times m \) orthogonal matrix, \( V \) is an \( n \times n \) orthogonal matrix, and \( \Sigma \) has the form

\[
\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}
\]

where \( \text{rk} A = r \) and \( \sigma_1, \ldots, \sigma_r \) are the singular values of \( A \).

To do the proof/construction of the SVD, we need the following result:

Lemma 3. The columns of \( U \) are orthonormal eigenvectors of \( A A^\top \), the columns of \( V \) are orthonormal eigenvectors of \( A^\top A \) and \( \sigma_i^2 \)'s are the eigenvalues of \( A A^\top \) (or \( A^\top A \)).

Consider: Why are the columns of \( U \) are orthonormal eigenvectors of \( AA^\top \)? How about the columns of \( V \) being orthonormal eigenvectors of \( A^\top A \)?
Problem 1. Find the SVD of

\[ A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \]

by first finding \( \Sigma \), then \( U \), then \( V \)
Consider. (Geometric Interpretation of SVD.) Consider the matrix

\[ A = \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix} \]

with SVD

\[ U = \begin{bmatrix} -0.8507 & -0.5257 \\ -0.5257 & 0.8507 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 8.7134 & 0 \\ 0 & 3.3282 \end{bmatrix}, \quad V = \begin{bmatrix} -0.5946 & 0.8041 \\ -0.8041 & -0.5946 \end{bmatrix} \]

How can we geometrically interpret the linear map \( A \) through its SVD? Consider the unit circle and let’s see how the matrix can transform it.

Let’s go step-by-step through how SVD decomposes this process into three transformations:

(a) Unit circle.
(b) \( V \) rotates.
(c) \( \Sigma \) scales.
(d) \( U \) rotates again.

Figure 1: Visual representation of linear map \( A \) acting on unit circle.

Figure 2: Visualization of SVD.
Problem 2. Matrix induced 2-norm. Prove that

\[ \|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \sigma_1 \]

Fact 1. (Relation between null, range, and SVD)

- The first \( k \) columns of \( U \) (first \( k \) left singular vectors) provide an orthonormal basis for the range of \( A \).
- The last \( n - k \) right singular vectors (the last \( n - k \) rows of \( V^T \)) provide an orthonormal basis for the null space of \( A \).


Problem 3. Suppose we have a constraint \( Ax = 0 \) with \( A \in \mathbb{R}^{m \times n} \) and \( x \in \mathbb{R}^n \). How can we come up with an expression for \( x \) in terms of \( A \) using the SVD?
2 Lipschitz Continuity

Definition 4. $f$ is **globally Lipschitz continuous** (LC) if there exists a piecewise continuous function $k(t)$ such that

$$
\|f(x, t) - f(y, t)\| \leq k(t)\|x - y\|
$$

for all $x, y \in \mathbb{R}^n$, for all $t \in \mathbb{R}^+$. 

$f$ is **locally Lipschitz continuous** in $U \subset \mathbb{R}^n$, if for every $x, y \in U$, the Lipschitz property above is satisfied.

**Consider.** Which norm should we use to check the Lipschitz condition?

Problem 4. (Local or global Lipschitz condition.) Consider the following system of differential equations:

- $\dot{x}_1 = x_1^2 + x_2^2$
- $\dot{x}_2 = x_1^2 - x_2^2$

Prove that this system is locally Lipschitz, but not globally Lipschitz.
• See "Examples" section of Lipschitz wiki page https://en.wikipedia.org/wiki/Lipschitz_continuity
• Review solving first order ODEs. "Paul's notes" has six scalar ($x \in \mathbb{R}^1$) examples: https://tutorial.math.lamar.edu/Classes/DE/Linear.aspx

3 Practice Prelim Problem

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Consider the set
$$A := \{x : Ax = b\}$$
where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ are given.

1. What is the dimension of $A$? Does it depend on $b$?

2. How would you determine a basis of the nullspace and range of $A$ numerically?

3. Assume $m = 1$, and let $A = a^T$, with $a \in \mathbb{R}^n$. Specify the answer to the previous question in that case.

4. We are given $N$ data points $x_i \in \mathbb{R}^n$, $i = 1, \ldots, N$. We would like to project these points on a hyperplane, so as to visualize the points on a single line. How would you choose the hyperplane? Discuss.