1 Second-Order Transfer Functions and Minimal Realizations

Consider a transfer function

$$Y(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} U(s)$$

The poles of this second-order system are at

$$s = -\xi\omega \pm \omega\sqrt{1 - \xi^2}$$

If we apply a step input $u(t)$ to the system ($U(s) = 1$), our output will look like

$$y(t)$$

Figure 1: $y(t)$ is in response to step change in $u(t)$, where $u(t)$ is the dotted line.

We see from the pole expression why $\xi$ is called the the damping ratio and $\omega$ is the natural frequency. If $\xi > 1$, the system is overdamped (no oscillations), $0 < \xi < 1$ the system is underdamped.

For more info: [http://www.controlsysacademy.com/0024/0024.html](http://www.controlsysacademy.com/0024/0024.html)

**Consider:** We’ve derived that you can convert a state space model to its associated transfer function with

$$H(s) = C(sI - A)^{-1}B + D$$. What about converting a transfer function to an associated state space model?

**Proposition 1:** Our minimal realization of a SISO system is always in controllable canonical form Why?

We can verify that

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & c_1 & \ldots & c_{n-1} \end{bmatrix}, \quad D = G(\infty)$$

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has the transfer function

$$H(s) = c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_0 + G(\infty)$$


2 Controllability and Observability

2.1 LTV systems

We derived in lecture the controllability and observability grammians

$$W_{G}(t_1, t_2) = \int_{t_1}^{t_2} \Phi(t, r)R(r)\Phi^T(t, r)dr$$

$$W_{B}(t_1, t_2) = \int_{t_1}^{t_2} \Phi(t, r)K(r)\Phi^T(t, r)dr$$

- Controllable on $[t_0, t_1] \iff \mathcal{R}(\mathcal{L}_u) = \mathcal{R}^n \iff \mathcal{R}(\mathcal{L}_C) = \mathcal{R}^n \iff \text{rank}(W_u) = n \iff x^T\mathbf{W}_u x > 0$

- Observable on $[t_0, t_1] \iff \mathcal{N}(\mathcal{L}_u) = \{0\} \iff \mathcal{N}(\mathcal{C}_y) = \{0\} \iff \text{rank}(W_y) = n \iff x^T\mathbf{W}_y x > 0$

2.2 LTI systems

Consider the Linear Time-Invariant system $\dot{x} = Ax + Bu$, $y = Cx + Du$

**Theorem 2.** The system is completely controllable on $[0, \Delta]$ for some $\Delta > 0$

$$\iff \text{rank} \begin{bmatrix} B \\ AB \\ \ldots \\ A^{n-1}B \end{bmatrix} = n \iff \text{rank} \begin{bmatrix} A - \xi I \\ \xi A - \xi I \end{bmatrix} = n \text{ for all } \xi \in \sigma(A)$$

**Theorem 3.** The system is completely observable on $[0, \Delta]$ for some $\Delta > 0$
Theorem 3. The system is completely observable on $[0, \Delta]$ for some $\Delta > 0$.

$$\implies \text{rank } \begin{bmatrix} C \\ CA \\ CA^{2} \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \implies \text{rank } \left[ \begin{bmatrix} A - \lambda I \end{bmatrix} \right] = n \text{ for all } s \in a(A)$$

$$\text{dim } \{ \{ A \} \} \times n \implies \text{dim } \{ \{ A \} \times n \}$$

Problem 1. Consider the controllability and observability Grammians $W_c, W_o$ of a linear time invariant system $(A, B, C)$ over the time period $[0, T]$.

a) Find expressions for $W_c, W_o$, which are the grammians after a similarity transform $z = Tz$, in terms of $A, B, C$, and $T$.

Bint: our transformed system is characterized by $T(A - \lambda I)T^{-1}, TB, TC^{-1})$

b) show that the eigenvalues of the product $M_1 = W_cW_o$ are equal to the eigenvalues of $M_2 = W_oW_c$

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3 Popov-Belevitch-Hautus (PBH) Test

Proposition 4. $\text{rank } \begin{bmatrix} A \\ B \\ \vdots \\ A^{n-1} \end{bmatrix} = n \text{ for all } s \in a(A) \implies \text{rank } \begin{bmatrix} B \\ A \\ \vdots \\ A^{n-1} \end{bmatrix} = n$

Proof: Reference: 221A Lecture 17, proving $(C) \implies (B)$

We define subspace $Q$ complete the column of $Q$

We found in $\text{rank } Q$. Using $\text{rank } a(Q) \leq n$.

so define subspace $V$ complete the column of $Q$

Ex: $J = \begin{bmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n \end{bmatrix}$

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Consider: Why are the PBH tests useful?

Definition 5. An LTI system is stable if all its uncontrollable modes are stable.

Equivalently:
\[ \text{rank} \begin{bmatrix} A & B \end{bmatrix} = n \quad \forall s \in \sigma(A) \cap \mathbb{C}_s \]

Definition 6. An LTI system is detectable if all of its unobservable modes are stable.

Equivalently:
\[ \text{rank} \begin{bmatrix} C \end{bmatrix} = n \quad \forall s \in \sigma(C) \cap \mathbb{C}_s \]

Consider: How are the definitions equivalent to their rank tests?

We can use the boolean algebra property \((x \implies y) \equiv \neg x \lor y\)

\[ a \implies b \implies c \lor b \implies c \implies a \lor b \]

Problem 2. (Prelim Fall 2018 Tamin)

The single input single output systems \( L_1 = \begin{bmatrix} A_1 & b_1 \end{bmatrix} \) and \( L_2 = \begin{bmatrix} A_2 & b_2 \end{bmatrix} \) are each completely controllable and completely observable.

Discuss the controllability and observability of the systems

\[
L_3 = \begin{bmatrix} A_1 & \alpha b_1 \\ 0 & A_2 \end{bmatrix}, \quad L_4 = \begin{bmatrix} A_1 & \beta b_1 \\ 0 & A_2 \end{bmatrix}
\]

in the two cases

(a) when \( A_1 \) and \( A_2 \) have no common eigenvalues;

(b) when \( A_1 \) and \( A_2 \) have at least one eigenvalue in common.

System 1:

\[ \text{a) assess controllability:} \]

\[ \text{b) assess observability:} \]

\[ \text{c) design matrix to implement desired controllability.} \]

Problem 3. For each of the following, provide either a proof or a counterexample:

(a) Suppose \((A, B)\) is controllable. Is the system \((A^T, B)\) controllable?

(b) Suppose \((A, B)\) is controllable. Is the system \((A, AB)\) controllable?

4 Controller and Observer Design

Problem 4 (Output feedback design). Consider the linear system defined by

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{align*}
\]

(a) Is the system controllable? Is it observable?

(b) Can the closed loop poles of the system be placed at \( \lambda_1 = -2, \lambda_2 = -3 \) using output feedback alone?
Problem 4 cont.
Now consider the same plant with an additional state measurement state such that
\[ y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x \]
(c) Is the system still controllable and observable?
(d) Can the closed loop poles of the system be placed at \( \lambda_1 = -2 \) and \( \lambda_2 = -2i \)?
(e) Explain how the closed loop poles of the system could be placed at \( \lambda_1 = -2 \) and \( \lambda_2 = -2i \) using only a single sensor, i.e., the output is one-dimensional.

Problem 5 (Observer design). Consider the linear system defined by
\[ \dot{x} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \]
\[ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \]
Can you design a Luenberger observer (full order observer) for this system, which has three poles at \(-2\)?

Consider: Kalman Filters (not covered in 221A) and, Luenberger observers determine estimates of the state \( x \) using the output of a system \( p \). What is the difference between them? (generation of state estimate)

Kalman filters are for stochastic linear system: \( z(k+1) = Ax(k) + Bu(k) + w(k) \)
\( y(k) = Cx(k) + v(k) \)
Kalman Filter: \( \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K(y(k) - C\hat{x}(k)) \)
Luenberger observer: \( \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + T(y(k) - C\hat{x}(k)) \)

⽌一 advance: equation is deterministic, not stochastic
- T computed, \( P \) updated at each step
- Kalman Filter yields estimate that is optimal (certain stochastic error matrix), called MSE

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**Problem 17**

Consider the dynamical model
\[ \dot{x} = \frac{1}{M} \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & -b & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f(t) \]
\[ \ddot{y} = \frac{1}{M} \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & -b & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \]
where \( M, a, b, f(t), y, u(t) \) are positive constants. This model describes the linearized equations of motion of an inverted pendulum where \( y(t) \) is the position of the cart, \( \ddot{y}(t) \) is the angle of the pendulum, and \( u(t) \) is the input force.
(a) Is it possible to design an asymptotically stabilizing state controller that uses only \( \dot{y}(t) \) measurements for feedback?
(b) What if we allow a dynamic controller?
(c) What about a dynamic controller that uses only \( y(t) \) measurements?
(d) How would you answer part c if \( a \ll M \)?
c) What about a dynamic controller that uses only $p(t)$ measurements?

d) How would you answer part c if $w < M$?

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(a) Explain whether state feedback changes controllability; that is, if the pair $(A, B)$ is controllable, can you claim the same for $(A + BK, B)$ where $K$ is an arbitrary state feedback matrix?

(b) Explain whether state feedback changes observability.

c) Explain whether adding an integrator of the input changes controllability; that is, if the pair $(A, B)$ is controllable, can you claim the same for the system $\dot{z} = Az + Bu + w$?

d) Two point masses move on a line as depicted in the figure below. Suppose the masses are unity, there is no friction, and a separate force control input is available for each mass, so that the dynamical model is:

\[ \dot{x}_i = x_{i-1} - x_i \]

Explain if a controller can be designed to move $d_i$ and $d_0$ to zero using the relative distance measurement $y = d_i - d_0$.