1 State space stability (internal stability)

Consider the LTV system $\dot{x}(t) = A(t)x(t)$ for the following definitions:

**Definition 1.** $x_e = 0$ is an equilibrium point of the system if and only if $A(t)x_e = 0 \ \forall t$.

**Consider.** What does stability have to do with equilibriums? Why are there so many types of stability to keep track of?

1. In control theory, we generally care about taking a system to an equilibrium eventually, and keeping it there. That's why we study stability around an equilibrium.
2. How fast, or the behavior of, the system as it goes to that equilibrium (if it ever does) defines different types of stability

**Consider.** For time-invariant $\dot{x} = Ax$, does there always exist an equilibrium of at zero. Is it the unique equilibrium?

**Definition 2.** $x_e = 0$ is (internally state) stable if and only if, $\forall x_0 \in \mathbb{R}^n, \forall t_0 \in \mathbb{R}_+$, the map $[t \to x(t) = \Phi(t,t_0)x_0]$ is bounded for all $t \geq t_0$.

**Theorem 3.** Consider the linear time invariant system $\dot{x}(t) = Ax(t)$. $x_0 = 0$ is stable if all the eigenvalues of $A$ are in the closed left half plane, and each of the jω-axis eigenvalues has a Jordan block of size 1.

**Definition 4.** $x_e = 0$ is asymptotically stable if and only if 1) $x_e = 0$ is stable and 2) $x(t) = \Phi(t,t_0)x_0$ tends to 0 as $t \to \infty$.

**Definition 5.** $x_e = 0$ is exponentially stable if and only if there exists $M, \alpha > 0$ such that $\|x(t)\| \leq Me^{-\alpha(t-t_0)}\|x_0\|$.

**Theorem 6.** Consider the linear time invariant system $\dot{x}(t) = Ax(t)$. $x_0 = 0$ is exponentially stable if and only if all of the eigenvalues of $A$ are in the open left half plane.
### Linear Time Varying Systems

**BIBO Stability**
- \( \exists k < \infty \text{ s.t. } \forall u \in L_\infty^n, \|y(t)\|_\infty \leq k\|u(t)\|_\infty \)
- \( \sup_{t \in \mathbb{R}} \left\{ \int_{-\infty}^{t} \|H(t, \tau)\|_{L_\infty} d\tau \right\} =: k < \infty \) & \( B(\cdot), C(\cdot), D(\cdot) \) are bounded
- Exponentially Stable & \( B(\cdot), C(\cdot), D(\cdot) \) are bounded

**Exponential Stability**
- \( \exists M, \alpha > 0 \text{ s.t. } \|x(t)\| \leq M \exp(-\alpha(t - t_0))\|x_0\| \)

**Asymptotic Stability**
- \( \exists M, \alpha > 0 \text{ s.t. } \|\Phi(t, t_0)\| \leq M \exp(-\alpha(t - t_0)) \)
- \( A(t) = A^T(t) \) and \( \sigma(A(t)) \leq -\mu \)
- \( \forall t \in \mathbb{R}, \mu \in \mathbb{R}_+ \)
- \( x_e = 0 \) is stable and \( t \to x(t) = \Phi(t, t_0)x_0 \) tends to 0 as \( t \to \infty \)

**State Space Stability**
- \( \forall x_0 \in \mathbb{R}^n, \forall t_0 \in \mathbb{R}^n, \) the map \( t \to x(t) = \Phi(t, t_0)x_0 \) is bounded \( \forall t \geq t_0 \)
- i.e. \( \exists M \text{ s.t. } \|\Phi(t, t_0)\| \leq M \forall t \geq t_0 \)

### Linear Time Invariant Systems

**BIBO Stability**
- \( \exists k < \infty \text{ s.t. } \forall u(\cdot) \in L_\infty^n, \|y(\cdot)\|_\infty \leq k\|u(\cdot)\|_\infty \)
- \( \sup_{t \in \mathbb{R}} \left\{ \int_{-\infty}^{t} \|G(\tau)\|_{L_\infty} d\tau \right\} =: k < \infty \) & \( B(\cdot), C(\cdot), D(\cdot) \) are bounded
- Exponentially Stable & \( B(\cdot), C(\cdot), D(\cdot) \) are bounded

**Exponential Stability**
- \( \sigma(A) \in \mathbb{C}_- \)

**Asymptotic Stability**
- \( \forall Q = Q^T > 0, \exists P = P^T > 0 \text{ s.t. } A^TP + PA = -Q \)
- where \( P \in \mathbb{R}^{n \times n}, Q \in \mathbb{R}^{n \times n} \) (Lyapunov)

**State Space Stability**
- \( \sigma(A) \in \mathbb{C}_- \) and each \( j\omega \)-axis eval has Jordan block of size 1
Problem 1. Under what conditions is the following system stable?

\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} x
\]

Problem 2. Consider the linear time varying system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t).
\end{align*}
\]  

Assume that the equilibrium \(0\) of \(\dot{x}(t) = A(t)x(t)\) is exponentially stable. Let \(B(\cdot), C(\cdot), D(\cdot)\) be bounded. Show that the system is BIBO stable.

2 Lyapunov Equation

Theorem 7. \(\dot{x} = Ax\) with \(A \in \mathbb{R}^{n \times n}\) is exponentially stable if and only if

\[
A^T P + PA = -Q
\]

has a unique solution \(P = P^T > 0\), for all \(Q = Q^T > 0\). \(P, Q \in \mathbb{R}^{n \times n}\).

Problem 3. Show that if \(\sigma(A) \subset \mathbb{C}_o\), then for given \(Q = Q^T > 0\) there exists a unique positive definite \(P = P^T\) solving

\[
A^T P + PA = -Q.
\]
Problem 4. Show that \( \dot{x}(t) = A(t)x(t) \) is stable if \( A(t) = -A(t)^T \).

3 Definition of Controllability and Observability

Consider a dynamical system \( D = (U, \Sigma, Y, s, r) \).

**Definition 8.** The system \( D \) is completely controllable (or just "controllable") on \([t_0, t_1]\) iff for any given \( x_1 \), for all \( x_0 \) there exists some \( u_{[t_0, t_1]} \in U \) that transfers \( x_0 \) to \( x_1 \)

**Remark 1.** In other words, \( \forall x_0 \in \Sigma \) the map \( s(t_1, t_0, x_0, u_{[t_0, t_1]}): U \to \Sigma \) is surjective.
**Definition 9.** The system $D$ is completely observable on $[t_0, t_1]$ iff for all $u_{[t_0, t_1]} \in \mathcal{U}$ and for all $y_{[t_0, t_1]} \in \mathcal{Y}$, $x_0$ is uniquely determined.

**Remark 2.** In other words, $\forall y_{[t_0, t_1]} \in \mathcal{Y}$ the map $\rho(t_1, t_0, x_0, u_{[t_0, t_1]}) : \Sigma \to \mathcal{Y}$ is injective.

**Problem 5. Is this system controllable? Observable?**

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 4 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad (2)$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u \quad (3)$$

**Problem 6.** a) Draw the feedback diagram for output feedback and b) derive the closed-loop dynamics matrix that results from applying linear output feedback to a linear system.

**Remark.** The definitions of observability and controllability are useful for conceptual questions and systems of certain structures, but it’s hard to assess whether a general large system is controllable and/or observable. That is why in lecture we are deriving a computational rank test for complete controllability (c.c.) and complete observability (c.o.)...
Consider the LTI system:

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (1)  \\
\[ y = Cx. \]  \hspace{1cm} (2)

1. Suppose there exists a matrix \( P = P^T > 0 \) and a constant \( \alpha \) such that:

\[ A^T P + PA < \alpha P. \]  \hspace{1cm} (3)

(a) Based on this inequality, which region in the complex plane can you conclude that the eigenvalues of \( A \) lie in?

(b) Suppose the matrix inequality (3) holds with \( \alpha = 0 \) and, in addition,

\[ PB = C^T. \]  \hspace{1cm} (4)

Show that the system (1)-(2) is asymptotically stable for any feedback:

\[ u = -ky \]  \hspace{1cm} (5)

with a nonnegative gain \( k \geq 0 \).

2. Suppose, instead of the inequality (3), the matrix \( P = P^T > 0 \) satisfies the equality:

\[ A^T P + PA = 0. \]  \hspace{1cm} (6)

(a) Determine the region in the complex plane where the eigenvalues of \( A \) lie in.

(b) Does (4) guarantee asymptotic stability for the feedback (5) with a positive gain \( k > 0 \)? If not, what additional conditions would you need?
Consider a continuous-time LTI system \( \dot{x}(t) = Ax(t), \ t \geq 0 \), with no input (such a system is said to be autonomous), and output \( y(t) = Cx \). We wish to evaluate the energy contained in the system’s output, as measured by the index

\[
J(x_0) := \int_0^\infty y(t)^T y(t) \, dt = \int_0^\infty x(t)^T Q x(t) \, dt,
\]

where \( Q := C^T C \succeq 0 \).

1. Show that if the system is stable, then \( J(x_0) < \infty \), for any given \( x_0 \). \textit{Hint:} show that \( \|y(t)\|_2 \leq c \|x_0\|_2 e^{\sigma_{max} t} \), where \( \sigma_{\text{max}} \) is the maximum real part of the eigenvalues \( \lambda_i \) of \( A \), and \( c > 0 \) is some constant.

2. Show that if the system is stable and there exist a matrix \( P \succeq 0 \) such that

\[
A^T P + PA + Q \preceq 0,
\]

then it holds that \( J(x_0) \leq x_0^T P x_0 \). \textit{Hint:} consider the quadratic form \( V(x(t)) = x(t)^T P x(t) \), and evaluate its derivative with respect to time.

3. Explain how to compute a minimal upper bound on the state energy, for the given initial conditions.
Professor Fearing  Linear Systems Prelim  Fall 2015

1. Stability
   • Given linear time invariant system $\dot{x} = Ax$.
   • Given a matrix $M$ is positive definite symmetric.
   • Given $V = x^T M x$, and $\dot{V} < 0$ for any trajectory.
   Show using these facts the possible range of eigenvalues for $A$.

2. State Equations
   Consider the continuous time linear system defined by:
   \[
   \begin{bmatrix}
   \dot{x}_1 \\
   \dot{x}_2
   \end{bmatrix}
   = \begin{bmatrix}
   -1 & 0 \\
   0 & -2
   \end{bmatrix}
   \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix}
   + \begin{bmatrix}
   1 \\
   2
   \end{bmatrix} u
   \]

   a. For $u(t) = 0$, sketch the state trajectory with:
      \[
      x_a(t = 0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad x_b(t = 0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
      \]

   b. For $u(t) = 1$ for $t \geq 0$, sketch the state trajectory with:
      \[
      x_a(t = 0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad x_b(t = 0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
      \]

   c. Given an initial condition $x_o = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, explain how would you find a $u(t)$ such that $x(t)$ asymptotically approaches a finite fixed value, e.g. $x_f = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

   d. Given an initial condition $x_o = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, explain if it is possible to find a $u(t)$ such that $x(t)$ asymptotically approaches $x_f = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ with fixed $x_1(t) = 2$ for $t \geq 0$?