Tips for doing proofs

- Don't be scared
- Writing things down!
  - Write precise definitions
  - Rewrite things in words in math notation
  - Simple example
    - In variables → try 2 variables
    - Variables → plug in some numbers
  - Work from start + end
    - What we know:  ∴ look for connections between them
    - What we want to show:
  - Write down definitions & or facts that might be related
- You should understand all the steps in your proof
  - Your reader should understand
  - Add some justification
Types of proofs: (that we’ve introduced)

- Direct proof: series of mathematical steps.
  ▲ most common in this class

- Constructive proofs:
  "Show that there exists…"
  - give an example of something that meets the requirements

- Proof by contradiction:
  "Show there does not exist…"
  ▲ does not exist

→ assuming ▲ does exist
→ use this assumption
→ find a contradiction
  ex) \( 10 = 0 \)
  ex) contradiction with a different assumption.
  \( \exists \vec{v}, \ldots \vec{v}_n \) are linearly dependent
  \( \exists \vec{v}, \ldots \vec{v}_n \) are linearly independent
Example 1

Let \( \tilde{x} \) be orthogonal to \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n \). Prove that \( \tilde{x} \) is orthogonal to any vector in the span \( \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\} \).

Proof

By the def. of span

\[ \tilde{v} \in \text{span} \{\tilde{a}_1, \ldots, \tilde{a}_n\} \]

if \( \tilde{v} = c_1 \tilde{a}_1 + c_2 \tilde{a}_2 + \ldots + c_n \tilde{a}_n \) is a representative vector in the span \( \{\tilde{a}_1, \ldots, \tilde{a}_n\} \)

Def of orthogonality

\[ \langle \tilde{x}, \tilde{v} \rangle = 0 \]

\[ \tilde{x}^T \tilde{v} = 0 \]

we also know

\[ \langle \tilde{x}, \tilde{a}_1 \rangle = 0 \]

\[ \tilde{x}^T \tilde{a}_1 = 0 \]

\[ \vdots \]

\[ \langle \tilde{x}, \tilde{a}_n \rangle = 0 \]

\[ \tilde{x}^T \tilde{a}_n = 0 \]

Try:

\[ \langle \tilde{x}, \tilde{v} \rangle = \langle \tilde{x}, c_1 \tilde{a}_1 + c_2 \tilde{a}_2 + \ldots + c_n \tilde{a}_n \rangle \]

\[ = \langle \tilde{x}, c_1 \tilde{a}_1 \rangle + \langle \tilde{x}, c_2 \tilde{a}_2 \rangle + \ldots + \langle \tilde{x}, c_n \tilde{a}_n \rangle \]

\[ = c_1 \langle \tilde{x}, \tilde{a}_1 \rangle + c_2 \langle \tilde{x}, \tilde{a}_2 \rangle + \ldots + c_n \langle \tilde{x}, \tilde{a}_n \rangle \]

\[ = 0 \]

we've shown \( \tilde{x} \) is orthogonal to \( \tilde{v} \). Since \( \tilde{v} \) represents every vector in \( \text{span} \{\tilde{a}_1, \ldots, \tilde{a}_n\} \) we've shown \( \tilde{x} \) is orthogonal to \( \text{span} \{\tilde{a}_1, \ldots, \tilde{a}_n\} \).
**Example 2**  [MT 1 Fall 2019]

**Thm** If \( A \) has a non-trivial nullspace, then \( A \) is not invertible.

**Proof**

**Given:** \( A \vec{v} = \vec{0} \quad \vec{v} \neq \vec{0} \) \( \Rightarrow \) \( \exists \)

\[ \text{Want to show: } A^{-1} \text{ does not exist} \]

\[ \Rightarrow \text{hard.} \]

**Assume** \( A^{-1} \) exists. \( \exists \) "know"

**Try:** \( A^{-1} A \vec{v} = A^{-1} \vec{0} \) \( \Leftarrow \) left mult. by \( A^{-1} \)

\[ \vec{v} = \vec{0} \quad \Leftarrow \text{doing math / def. of inverse} \]

Give a contradiction.

**Therefore** \( A^{-1} \) must not exist.

---

Contra positive

- If \( p \) then \( q \) equivalent to
- If not \( q \) then not \( p \)

If \( A \) is invertible then

\( A \) has a trivial nullspace.
Example 3

If \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \) and \( \vec{u} + \vec{v} + \vec{w} = \vec{0} \), prove \( \text{span} \{ \vec{u}, \vec{v}, \vec{w} \} = \text{span} \{ \vec{u}, \vec{v}, \vec{w} \} \).

1. If \( \vec{x} \in \text{span} \{ \vec{u}, \vec{v}, \vec{w} \} \) then \( \vec{x} \in \text{span} \{ \vec{u}, \vec{v}, \vec{w} \} \).
2. If \( \vec{y} \in \text{span} \{ \vec{u}, \vec{v}, \vec{w} \} \) then \( \vec{y} \in \text{span} \{ \vec{u}, \vec{v}, \vec{w} \} \).

Aside:
"if and only if"

→ prove both directions

1. \( \vec{x} = \alpha_1 \vec{u} + \alpha_2 \vec{v} \) for some \( \alpha_1, \alpha_2 \).

2. If \( \vec{u} = -\vec{v} - \vec{w} \),

   \[ \vec{x} = \alpha_1 (-\vec{v} - \vec{w}) + \alpha_2 \vec{v} \]

   \[ \vec{x} = -\alpha_1 \vec{w} - \alpha_1 \vec{v} + \alpha_2 \vec{v} \]

   \[ \vec{x} = (\alpha_2 - \alpha_1) \vec{v} - \alpha_1 \vec{w} \]

   Let \( \beta_1 = \alpha_2 - \alpha_1 \)

   \( \beta_2 = -\alpha_1 \)

   Want to show: \( \vec{x} = \beta_1 \vec{u} + \beta_2 \vec{w} \) for some \( \beta_1, \beta_2 \).

2. \( \vec{y} = \beta_1 \vec{v} + \beta_2 \vec{w} \) for some \( \beta_1, \beta_2 \).

   \( \vec{w} = -\vec{u} - \vec{v} \)

   \[ \vec{y} = \beta_1 \vec{v} + \beta_2 (-\vec{u} - \vec{v}) \]

   \[ \vec{y} = \beta_1 \vec{v} + \beta_2 \vec{u} + \beta_2 \vec{v} \]

   \[ \vec{y} = -\beta_2 \vec{u} + (\beta_1 - \beta_2) \vec{v} \]

   Let \( \alpha_1 = -\beta_2 \)

   \( \alpha_2 = \beta_1 - \beta_2 \)

   Want to show: \( \vec{y} = \alpha_1 \vec{u} + \alpha_2 \vec{v} \) for some \( \alpha_1, \alpha_2 \).
Example 4

Let \( P, Q \in \mathbb{R}^{n \times n} \) be square.

If \( Q \) has rank \( n \) and \( PQ = 0 \) (where \( 0 \) is a matrix \( (n \times n) \) of all zeros), prove that \( P \) must be all zeros.

**Proof**

Since \( Q \) is full rank, then we know \( Q \) is invertible.

\[
PQ Q^{-1} = OQ^{-1} \quad \text{left mult. by } Q^{-1}
\]

\[
P = 0
\]

Let columns of \( Q \) be \( \vec{q}_1, ..., \vec{q}_n \)

we know by def. of mat. multiplicaton

\[
P\vec{q}_i = \vec{0} \quad \alpha_1 P\vec{q}_1 + \alpha_2 P\vec{q}_2 + ... + \alpha_n P\vec{q}_n = \vec{0}
\]

\[
\text{P} \text{ is full rank, then}
\]

\[
P (\alpha_1 \vec{q}_1 + ... + \alpha_n \vec{q}_n) = \vec{0}
\]

since \( \text{span} \{ \vec{q}_1, ..., \vec{q}_n \} = \mathbb{R}^n \)

\[
P\vec{x} = \vec{0} \quad \text{for any } \vec{x}
\]

Assume \( P \neq 0 \). Then there is at least one column \( P_i \neq 0 \). If \( \vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \) in position

\[
P\vec{x} = \vec{P}_i \quad \text{contradiction with } P\vec{x} = \vec{0} \quad \text{for all } \vec{x}
\]

so \( P = 0 \).