EECS 16A  
Module 3  
Topics  
Classification of signal  
Estimation of signal delay

* HW6A will be up today

Problems covering lec 6A

* Last lecture: Norm, inner product  
Classification

Norm: \[ r = \text{Norm of } \mathbf{u} = ||\mathbf{u}|| = \sqrt{u_1^2 + \cdots + u_n^2} \]

Inner product:

Inner product of \( \mathbf{u} \) & \( \mathbf{v} \) = \( \langle \mathbf{u}, \mathbf{v} \rangle \)
\[ = u_1v_1 + \cdots + u_nv_n \]
\[ = \mathbf{u}^\intercal \mathbf{v} \quad \rightarrow \text{transpose} \]

* \( \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \)

* \( \langle \mathbf{u}, \mathbf{u} \rangle = u_1^2 + \cdots + u_n^2 = ||\mathbf{u}||^2 \)
Database:
\[ \tilde{s}_1 = [1 1 -1 -1 1 1]^T \quad ||\tilde{s}_1|| = \sqrt{6} \]
\[ \tilde{s}_2 = [1 -1 -1 1 1 1]^T \quad ||\tilde{s}_2|| = \sqrt{6} \]

Received:
\[ \tilde{r} = [1 1 -1 -1 1 1]^T = \tilde{s}_1 \]

Method 1: \[ \tilde{e}_1 = \tilde{r} - \tilde{s}_1, \quad ||\tilde{e}_1|| = 0 \] minimum
\[ \tilde{e}_2 = \tilde{r} - \tilde{s}_2, \quad ||\tilde{e}_2|| > 0 \]
Smaller \( ||\tilde{e}|| \) → better match

Method 2:
\[ \langle \tilde{r}, \tilde{s}_1 \rangle = 1 + 1 + 1 + 1 + 1 + 1 = 6 = ||\tilde{s}_1||^2 \]
\[ \langle \tilde{r}, \tilde{s}_2 \rangle = 1 - 1 + 1 - 1 + 1 + 1 = 2 < 6 \]
Bigger inner product → better match.
Relationship between inner product & error

\[ \text{error, } \bar{e} = \bar{r} - \bar{s} \]
\[ \| e \|_2^2 = \langle \bar{e}, \bar{e} \rangle = \bar{e}^T \bar{e} \]
\[ = (\bar{r} - \bar{s})^T (\bar{r} - \bar{s}) \]
\[ = (\bar{r}^T - \bar{s}^T) (\bar{r} - \bar{s}) \]
\[ = \bar{r}^T \bar{r} + \bar{s}^T \bar{s} - \bar{s}^T \bar{r} - \bar{r}^T \bar{s} \]
\[ = \| \bar{r} \|^2 + \| \bar{s} \|^2 - \langle \bar{s}, \bar{r} \rangle - \langle \bar{r}, \bar{s} \rangle \]
\[ = \| \bar{r} \|^2 + \| \bar{s} \|^2 - 2 \langle \bar{r}, \bar{s} \rangle \]

When \( \langle \bar{r}, \bar{s} \rangle \rightarrow \max \)
\[ \| \bar{e} \|_2 \rightarrow \min \]

Ex: \( \bar{s}_1 = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}^T \)
\[ \| \bar{s}_1 \| = \sqrt{2} \]
\( \bar{s}_2 = \begin{bmatrix} 4 & -4 & -4 & 4 & 4 & 4 \end{bmatrix}^T \)
\[ \| \bar{s}_2 \| = 4 \sqrt{2} \]
\( \bar{r} = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}^T = \bar{s}_2 \)
\[ \langle \bar{r}, \bar{s}_1 \rangle = 6 = \| \bar{s}_1 \|^2 \]
\[ \langle \bar{r}, \bar{s}_2 \rangle = 4 - 4 + 4 - 4 + 4 + 4 = 8 \]
Prob: \( \bar{r} \) matches \( s_1 \), but \( \langle \bar{r}, \bar{s}_2 \rangle \) is larger.

Solution:

Design database signals with the same norm. \( \| s_1 \| = \| s_2 \| \)

HW: shazam problem

Sending two signals simultaneously:

\[
\bar{r} = \bar{s}_1 + \bar{s}_2 + \bar{n} \quad \bar{n} = \text{noise} \\
\| \bar{r} \| = \| \bar{s}_2 \| \\
\langle \bar{r}, \bar{s}_1 \rangle = \langle \bar{s}_1 + \bar{s}_2 + \bar{n}, \bar{s}_2 \rangle \\
= \langle \bar{s}_1, \bar{s}_1 \rangle + \langle \bar{s}_2, \bar{s}_1 \rangle + \langle \bar{n}, \bar{s}_1 \rangle \\
= \| \bar{s}_1 \|^2 + \langle \bar{s}_2, \bar{s}_1 \rangle + \langle \bar{n}, \bar{s}_1 \rangle \\
\text{\underline{Interference}}
\]

\[
\langle \bar{s}_2, \bar{s}_1 \rangle = 113 \| \| \bar{s}_1 \| \cos (\theta_2 - \theta_1) \rightarrow \text{lec 5D} \\
\text{\underline{Minimize}}
\]

Choose \( \theta_2 - \theta_1 = 90^\circ \)

so that \( \langle \bar{s}_2, \bar{s}_1 \rangle = 0 \)
i.e. make $\tilde{S}_1$ & $\tilde{S}_2$ orthogonal

$$\langle \tilde{n}, \tilde{S}_1 \rangle = \text{small if } n \text{ is random noise}$$

$$\langle \tilde{r}, \tilde{S}_1 \rangle = \| \tilde{S}_1 \|^2$$

$$\langle \tilde{r}, \tilde{S}_2 \rangle = \| \tilde{S}_2 \|^2 + \langle \tilde{S}_2, \tilde{S}_1 \rangle + \langle \tilde{n}, \tilde{S}_2 \rangle$$

$$\leq \| \tilde{S}_2 \|^2 \quad \text{Small}$$

Requirements for database:

* $\| \tilde{S}_1 \| = \| \tilde{S}_2 \|$

* $\tilde{S}_1$ & $\tilde{S}_2$ are orthogonal

$$\tilde{S}_1 = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}^T$$

$$\tilde{S}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}^T$$

$$\langle \tilde{S}_1, \tilde{S}_2 \rangle = 0$$

$$\| \tilde{S}_1 \| = \| \tilde{S}_2 \| = 6$$

If $\tilde{r} = \begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}^T = \tilde{S}_2$

$$\langle \tilde{r}, \tilde{S}_1 \rangle = \langle \tilde{S}_2, \tilde{S}_1 \rangle = 0$$

$$\langle \tilde{r}, \tilde{S}_2 \rangle = \langle \tilde{S}_2, \tilde{S}_2 \rangle = \| \tilde{S}_2 \|^2 = 6$$
**Case I**

\[ r = \tilde{s}_1 + \tilde{s}_2 \]

\[ \langle \tilde{r}, \tilde{s}_1 \rangle = 6 \checkmark \]

\[ \langle \tilde{r}, \tilde{s}_2 \rangle = 6 \checkmark \]

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**Case II**

\[ \tilde{r} = \tilde{s}_1 \]

\[ \langle \tilde{r}, \tilde{s}_1 \rangle = 6 \checkmark \]

\[ \langle \tilde{r}, \tilde{s}_2 \rangle = 0 \]

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**Case III**

\[ \tilde{r} = \tilde{s}_2 \]

\[ \langle \tilde{r}, \tilde{s}_1 \rangle = 0 \]

\[ \langle \tilde{r}, \tilde{s}_2 \rangle = 6 \checkmark \]

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**Toy GPS problem:**

\[ s_1 = [1 \ 1 \ -1 \ -1 \ 1 \ 1]^T \quad ||\tilde{s}_1|| = ||\tilde{s}_2|| \]

\[ s_2 = [1 \ 1 \ -1 \ 1 \ -1 \ 1]^T \quad \langle \tilde{s}_1, \tilde{s}_2 \rangle = 0 \]

**Distance:** \( d \)

**Signal velocity:** \( v \)

**Signal delay, \( t_d = d/v \)**

\[ d = vt_d \]

\[ S_1(t) \]

\[ R(t) \]
From the plot:

\[
\begin{align*}
    s_1[0] &= 1 = r[5] \\
    s_1[1] &= 1 = r[6] \\
    s_1[3] &= -1 = r[8] \\
    s_1[5] &= 1 = r[10]
\end{align*}
\]

Inner product:

\[
\begin{align*}
    \tilde{r}[7] &= \left[ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \right] \\
    \tilde{s}_1[t] &= \left[ 1 \ 1 \ -1 \ -1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \right]
\end{align*}
\]

\[
\langle \tilde{r}[t], \tilde{s}_1[t] \rangle = 1 \neq 6 \text{ i.e.} \|\tilde{s}_1\|^2
\]

\[
\begin{align*}
    \tilde{r}[t] &= \left[ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \right] \\
    \tilde{s}_1[t-5] &= \left[ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1 \ -1 \ 1 \ 1 \right] \\
    \langle \tilde{r}[t], \tilde{s}_1[t-5] \rangle &= \zeta = \|\tilde{s}_1\|^2
\end{align*}
\]
How do we know delay is 5
Find \( \langle r[t], \bar{s}_1[t-k] \rangle \) for varying \( k \)
where \( k \) is the time shift of \( \bar{s}_1[k] \)

Powerpoint demo of sliding inner product

\[
\langle \tilde{r}[t], \bar{s}_1[t-k] \rangle = \sum_{i=-\alpha}^{\alpha} r[i] \bar{s}_1[i-k] \rightarrow \text{generic expression for } -\alpha \text{ to } \alpha
\]

= cross-correlation of \( \bar{s}_1 \) w.r.t. \( \bar{r} \)

= \( \text{corr}_r(s_1)[k] \)

\[
\text{corr}_r(s_1)[k] = \sum_{i=-\alpha}^{\alpha} r[i] s_1[i-k]
\]

\[
= \langle \bar{r}[t] \bar{s}[t-k] \rangle
\]

Dis 6A, 6B: mech prob