1. Steady and Unsteady States

(a) You’re given the matrix \( M \):

\[
M = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 1 & -2 \\
0 & 0 & 2
\end{bmatrix}
\]

Which generates the next state of a physical system from its previous state: \( \vec{x}[k + 1] = M \vec{x}[k] \). (\( \vec{x} \) could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span(\( \vec{v}_1 \)), associated with \( \lambda_1 = 1 \)

ii. span(\( \vec{v}_2 \)), associated with \( \lambda_2 = 2 \)

iii. span(\( \vec{v}_3 \)), associated with \( \lambda_3 = \frac{1}{2} \)

(b) Define \( \vec{x} = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 \), a linear combination of the eigenvectors. For each of the cases in the table, determine if \( \lim_{n \to \infty} M^n \vec{x} \) converges. If it does, what does it converge to?

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>Converges?</th>
<th>( \lim_{n \to \infty} M^n \vec{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
</tr>
<tr>
<td>( \neq 0 )</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
</tr>
<tr>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
</tr>
<tr>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
</tr>
</tbody>
</table>

2. Polynomials as a Vector Space

Let \( \mathbb{P}_2 \) be the set of polynomials of degree of at most two (that is, \( p(t) = at^2 + bt + c \)).

(a) Give a basis for \( \mathbb{P}_2 \).

(b) Consider the linear transformations

\[
T_1(f(t)) = 2f(t) \\
T_2(f(t)) = f'(t)
\]

For each, find the transformation matrix with respect to the basis from part (a).
(c) Suppose that \( \{ x_0, x_1, x_2 \} \) form a basis for \( \mathbb{P}_2 \) and that the following polynomials have the corresponding coordinates in this basis.

\[
\begin{align*}
(1, 1, 1) & \Rightarrow 2t^2 + 3t \\
(1, 0, -1) & \Rightarrow t + 1 \\
(0, 2, 0) & \Rightarrow 4t + 2
\end{align*}
\]

Find the basis vectors \( x_0, x_1, x_2 \).