Outline

• Wrap-up of previous lecture

• **Dynamic response** (chapter 3)
  – The effect of zeros and additional poles (3.5)
  – Stability (3.7)
    • Routh’s stability criterion
  – Models from data (3.8)

• **Basic Properties of Feedback** (chapter 4)
  – Basic equations of control (4.1)
Problem 3.26

3.26. Suppose you are to design a unity feedback controller for a first-order plant depicted in Fig. 3.50. (As you will learn in Chapter 4, the configuration shown is referred to as a proportional-integral controller.) You are to design the controller so that the closed-loop poles lie within the shaded regions shown in Fig. 3.51.

(a) What values of $\omega_n$ and $\zeta$ correspond to the shaded regions in Fig. 3.51? (A simple estimate from the figure is sufficient.)

(b) Let $K_p = \alpha = 2$. Find values for $K$ and $K_I$ so that the poles of the closed-loop system lie within the shaded regions.

(c) Prove that no matter what the values of $K_a$ and $\alpha$ are, the controller provides enough flexibility to place the poles anywhere in the complex (left-half) plane.
Effect of zeros and additional poles

- If system is too slow ($t_r$ large) we must raise natural freq ($\omega_n$), if transient has too much overshoot damping needs to be increased, if transient is too long poles need to be moved to the left in the s-plane…

- **Extra poles:** The major effect is an increase in the rise time

- **So what do the zeros do?**
  - they modify the coefficients of the exponential term whose shape is decided by the poles
  
  ➔ A zero near a pole reduces the amount of that term in the total response
Effect of zeros and additional poles

e.g.

\[ H_1(s) = \frac{2}{(s + 1)(s + 2)} = \frac{2}{s + 1} - \frac{2}{s + 2} \]

\[ H_2(s) = \frac{2(s + 1.1)}{(s + 1)(s + 2)} = \frac{0.18}{s + 1} + \frac{1.64}{s + 2} \]

Dramatic reduction (0.18) of coefficient due to the zero at \( s = -1.1 \)
Effect of zeros and additional poles

• Additional zero
Effect of zeros and additional poles

- Additional pole
Effect of zeros and additional poles

• Problem 3.32

3.32. Consider the system shown in Fig. 3.55, where

\[ G(s) = \frac{1}{s(s + 3)} \quad \text{and} \quad D(s) = \frac{K(s + z)}{s + p}. \]  \hspace{1cm} (3.82)

Find \( K, z, \) and \( p \) so that the closed-loop system has a 10\% overshoot to a step input and a settling time of 1.5 sec (1\% criterion).
Summary

Effects of Pole-Zero Patterns on Dynamic Response

1. For a second-order system with no finite zeros, the transient response parameters are approximated as follows:

   \[ t_r \approx \frac{1.8}{\omega_n}, \]

   Overshoot: \[ M_p \approx \begin{cases} 
   5\%, & \zeta = 0.7, \\
   16\%, & \zeta = 0.5, \\
   35\%, & \zeta = 0.3, 
\end{cases} \] (see Fig. 3.21),

   Settling time: \[ t_s \approx \frac{4.6}{\sigma}. \]

2. A zero in the left half-plane (LHP) will increase the overshoot if the zero is within a factor of 4 of the real part of the complex poles. A plot is given in Fig. 3.25.

3. A zero in the right half-plane will depress the overshoot (and may cause the step response to start out in the wrong direction).

4. An additional pole in the left half-plane will increase the rise time significantly if the extra pole is within a factor of 4 of the real part of the complex poles. A plot is given in Fig. 3.30.
Stability

• A system is stable if its initial conditions decay to zero and unstable if they diverge.

• A LTI system is stable if all the roots of the transfer function denominator polynomial (i.e. the poles) have negative parts (i.e. they are all in the LHP, $\sigma<0$), and is unstable otherwise.

  LTI characteristic equation:  
  \[ s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = 0 \]

  \[
  y(t) = \sum_{i=1}^{n} K_i e^{p_i t}, \quad \text{where } \{p_i\} \text{ are the roots and } \{K_i\} \text{ depend on initial conditions and zero locations}
  \]

• The system is stable **if and only if** (necessary and sufficient condition) every term goes to zero as \( t \to \infty \)

  \[ e^{p_i t} \to 0 \text{ for all } p_i, \text{ if all poles are in the LHP where } \text{Re}\{p_i\} < 0 \]

  – This is called **internal stability**
Stability

• If any poles in the RHP, system is **unstable**
  – Hence the \( j\omega \) axis is the stability boundary between asymptotically stable and unstable response

• If the system has nonrepeated \( j\omega \) axis poles \( \rightarrow \) **neutrally stable**
  – e.g. 1 pole at origin (an integrator) = non-decaying transient
  – e.g. 1 pair of complex \( j\omega \) axis poles = oscillating response with constant amplitude

• If system has repeated poles on the \( j\omega \) axis \( \rightarrow \) **unstable**
  – This is because of the term \( t \) multiplying the exponential in

\[
te^{\pm j\omega_i t}
\]
Ruth’s Stability Criterion

• Great method for finding out about the stability of a system without actually solving for the roots of the polynomial.
  – From the 19th century! (no computers, no MATLAB)

• Fast method, so is still useful for estimating the ranges of coefficients of polynomials for stability

• Considering the characteristic equation of an $n^{th}$ order system

$$a(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + ... + a_{n-1}s + a_n$$

• **Necessary (but not sufficient) condition** for stability is that all the coefficients $\{a_i\}$ of the characteristic polynomial be positive
  – missing (zero) or negative coefficients implies poles outside LHP
Ruth’s Stability Criterion

- **Ruth’s method**: computation of a triangular array function of \( \{a_i\} \)
- **Necessary and sufficient** condition for stability:
  - A system is stable if and only if *all* the elements in the first column of the Routh array are positive.

<table>
<thead>
<tr>
<th>Row</th>
<th>(n)</th>
<th>(s^n:)</th>
<th>1</th>
<th>(a_2)</th>
<th>(a_4)</th>
<th>(\ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>(n-1)</td>
<td>(s^{n-1}:)</td>
<td>(a_1)</td>
<td>(a_3)</td>
<td>(a_5)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>Row</td>
<td>(n-2)</td>
<td>(s^{n-2}:)</td>
<td>(b_1)</td>
<td>(b_2)</td>
<td>(b_3)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>Row</td>
<td>(n-3)</td>
<td>(s^{n-3}:)</td>
<td>(c_1)</td>
<td>(c_2)</td>
<td>(c_3)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>\vdots</td>
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<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>Row</td>
<td>2</td>
<td>(s^2:)</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row</td>
<td>1</td>
<td>(s^1:)</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Row</td>
<td>0</td>
<td>(s^0:)</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix} & = \frac{a_1 a_2 - a_3}{a_1} \\
\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix} & = \frac{a_1 a_4 - a_5}{a_1} \\
\det \begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix} & = \frac{a_1 a_6 - a_7}{a_1} \\
\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix} & = \frac{b_1 a_3 - a_1 b_2}{b_1} \\
\vdots
\end{align*}
\]

- You can multiply or divide an entire row by a constant for simplification
- **If all the elements in the 1st column are positive, then all the roots are in the LHP**
- **If not** all are positive \(\rightarrow\) \(N\) roots in RHP = \(N\) sign changes in column \((+-+ = 2\) changes)
Ruth’s Stability Criterion

3.38. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh’s stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) \( KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)} \)

(b) \( KG(s) = \frac{2(s+4)}{s^2(s+1)} \)
Ruth’s Stability Criterion

3.39. Use Routh’s stability criterion to determine how many roots with positive real parts the following equations have:

(a) \( s^4 + 8s^3 + 32s^2 + 80s + 100 = 0 \).

(b) \( s^4 + 2s^3 + 7s^2 - 2s + 8 = 0 \).
Example 3.29. Stability versus parameter range

Determine the range of K over which the system is stable

Solution: K > 7.5

Using Matlab we calculate the roots

- If K=7.5, roots at -5, 0±22j, system neutrally stable (presence of poles in the jω axis predicted by Routh’s method!)
- If K=13, roots at -4.06, -0.47±1.7j → System stable
- If K=25, roots at -1.90, -1.54±3.27j
Ruth’s Stability Criterion

- **Example 3.30.** Stability versus 2 parameter ranges
  - Determine the range of controller gain $K$, $K_I$ over which the system is stable

  ![Control System Diagram](image)

  - Solution: For internal stability we must have $K_I > 0$ and $K > K_I/3 - 2$

  ![Graph of Allowable Region](image)

  Allowable region for stability

  **Transient responses**

  Large steady-state error when $K_I = 0$

  ![Transient Response Graph](image)

  Proportional Integral controller
  (chapter 4)
Ruth’s Stability Criterion

• **Special case 1**: If only the first element if one of the rows is zero
  – Replace zero with small positive constant $\varepsilon$
  – Apply stability criterion by taking the limit as $\varepsilon \to 0$

\[
a(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9
\]

\[
\begin{array}{ccc}
s^5 & : & 1 \quad 2 \quad 6 \\
s^4 & : & 3 \quad 6 \quad 9 \\
s^3 & : & 0 \quad 3 \quad 0 \\
\text{New } s^3 \\
s^3 & : & \varepsilon \quad 3 \quad 0 \quad \leftarrow \text{Replace 0 by } \varepsilon \\
s^2 & : & \frac{2\varepsilon - 3}{\varepsilon} \quad 3 \quad 0 \\
s & : & 3 - \frac{3\varepsilon^2}{2\varepsilon - 3} \quad 0 \quad 0 \\
s^0 & : & 3 \quad 0
\end{array}
\]

2 sign changes: +++-++
2 poles **not** in the LHP
Ruth’s Stability Criterion

- **Special case 2**: When an entire row of the Routh array is zero
  - Form auxiliary equation from previous non-zero row
    \[ a_1(s) = \beta_1 s^{i+1} + \beta_2 s^i - 1 + \beta_3 s^{i-3} + \ldots \]
  - Where \( \{\beta_i\} \) of the \((i+1)\)th row in the array
  - Replace \( i \)th row by the coefficients of the derivative of the auxiliary polynomial
    \[ a(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 \]

\[
\begin{array}{ccc}
  s^5 & 1 & 11 & 28 \\
  s^4 & 5 & 23 & 12 \\
  s^3 & 6.4 & 25.6 & 0 \\
  s^2 & 3 & 12 & \\
  s & 0 & 0 & \leftarrow a_1(s) = 3s^2 + 12 \\
  \text{Calculate new } s \\
\end{array}
\]

No changes in 1st column

\[
\begin{array}{cc}
  s & 6 & 0 & \leftarrow \frac{da_1(s)}{ds} = 6s \\
  s^0 & 12 & \\
\end{array}
\]
Models from Experimental Data

• Why?
  – The best model built from equations of motion is still an approximation of reality!
    • Still, if model is accurate, need to verify with real data
  – When model is too complicated, or underlying physics are poorly understood
    • experimental data is necessary! (e.g. the brain)
  – When the environment of the system changes
    • Controller needs to be retuned based on experimental data. (e.g. real robots vs. simulated robots)
Models from Experimental Data

• Four kinds of experimental data for modeling
  – **Transient response.** Quick and easy to obtain but requires special tests to get enough SNR
  – **Frequency response.** Simple to obtain but time consuming and expensive. More in Ch. 6
  – **Stochastic steady-state information.** Cheap and easy to obtain, but quality is inconsistent resulting in incomplete models (e.g. the brain again!)
  – **Pseudorandom-noise.** Generated in digital computers. Analogous to broadband random signal
Basic equations of control
(section 4.1)

• Major goal of control design: to keep the error small for any input and in the face of unexpected parameter changes.

• After generating the model, we are going to deal with what it is that the control is required to do

• Control
  – **Open loop**: simple, no need of sensor, doesn’t introduce stability problems
  – **Closed-loop** (feedback control): complex, susceptibel to stability problems, but can give much better performance
    • Only option for naturally (open loop) unstable processes

• We need a language (equations) to describe the objectives
Basic equations of control

\[ Y_{ol} = H_r D_{ol} G R + G W \]
\[ E_{ol} = R - Y_{ol} = [1 - H_r D_{ol} G] R = [1 - T_{ol}] R - G W \]

\[ Y_{cl} = \frac{DG}{1 + DG} R + \frac{G}{1 + DG} W - \frac{DG}{1 + DG} V \]
\[ U = \frac{D}{1 + DG} R - \frac{DG}{1 + DG} W - \frac{D}{1 + DG} V \]
\[ E_{cl} = R - \left[ \frac{DG}{1 + DG} R + \frac{G}{1 + DG} W - \frac{DG}{1 + DG} V \right] = \frac{1}{1 + DG} R - \frac{G}{1 + DG} W + \frac{DG}{1 + DG} V \]
Basic equations of control

\[ Y_{cl} = \frac{DG}{1+DG} R + \frac{G}{1+DG} W - \frac{DG}{1+DG} V \]

\[ U = \frac{D}{1+DG} R - \frac{DG}{1+DG} W - \frac{D}{1+DG} V \]

\[ E_{cl} = R - \left[ \frac{DG}{1+DG} R + \frac{G}{1+DG} W - \frac{DG}{1+DG} V \right] = \]

\[ = \frac{1}{1+DG} R - \frac{G}{1+DG} W + \frac{DG}{1+DG} V \]

\[ S = \frac{1}{1+DG} \quad \text{Sensitivity function} \]

\[ T = 1 - S = \frac{DG}{1+DG} \quad \text{Complementary sensitivity function} \]

\[ E_{cl} = SR - SGW + TV \]