Notes 21 largely plagiarized by %khc

1 Notation

Getting through this section is going to require some notation:

CT	continous time
DT	discrete time
FT	Fourier transform
DTFT	discrete time Fourier transform
T	sampling period
$x(t)$	CT signal
$s(t)$	impulse train
$x_s(t)$	CT representation of DT signal $x[n]$
$x(nT)$	sequence, DT signal derived from $x(t)$, contains same information as $x[n]$, but with sampling period T included
$x[n]$	sequence, DT signal derived from $x(t)$, contains same information as $x(nT)$, but with sampling period T surpressed
$X(\omega)$	Fourier transform of $x(t)$
$S(\omega)$	Fourier transform of impulse train
$X(e^{j\omega T})$	discrete time Fourier transform of $x[n]$ or $x(nT)$, same shape as $X(e^{j\Omega})$, but with different ordinate scaling
$X(e^{j\Omega})$	discrete time Fourier transform of $x[n]$ or $x(nT)$, same shape as $X(e^{j\omega T})$, but with different ordinate scaling

2 Sampling

In the real world, when we sample, we take a CT signal $x(t)$ into a switch that is closed at time increments $t = nT$, as in Figure 1(a). We call the output of this switch $x(nT)$, a sequence derived from $x(t)$. The values of this sequence satisfy the relation:

$$x(nT) = x(t)|_{t=nT}$$

We can surpress this dependence on the sampling period T by thinking of $x(nT)$ being indexed on the integer n. This then gives us the sequence $x[n]$, which contains the same information as in $x(nT)$, with the exception of the sampling period. Because $x[n]$ is a sequence, we can take its DTFT, giving us $X(e^{j\Omega})$. Or we could take the DTFT of $x(nT)$ instead, which gives us $X(e^{j\omega T})$. Note that there is little difference between the two. $X(e^{j\Omega})$ has been normalized so that there is no time dependence; it is periodic with period 2π. $X(e^{j\omega T})$ has not been normalized; it retains the time dependence, and is consequently periodic with period $\frac{2\pi}{T}$.

Exercise Verify this.

But how do we relate this DTFT with $X(\omega)$, the FT of the original CT signal $x(t)$? Let's use a math trick and consider the setup in Figure 1(b), which says for us to take the CT $x(t)$ and multiply it by an impulse train $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

Note that you wouldn't do this in real life. *This is only a math trick to try to figure out how to relate $X(e^{j\omega T})$ and/or $X(e^{j\Omega})$ with $X(\omega)$.*

The product is then:

$$\begin{aligned} x_s(t) &= x(t)s(t) \\ &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned}$$

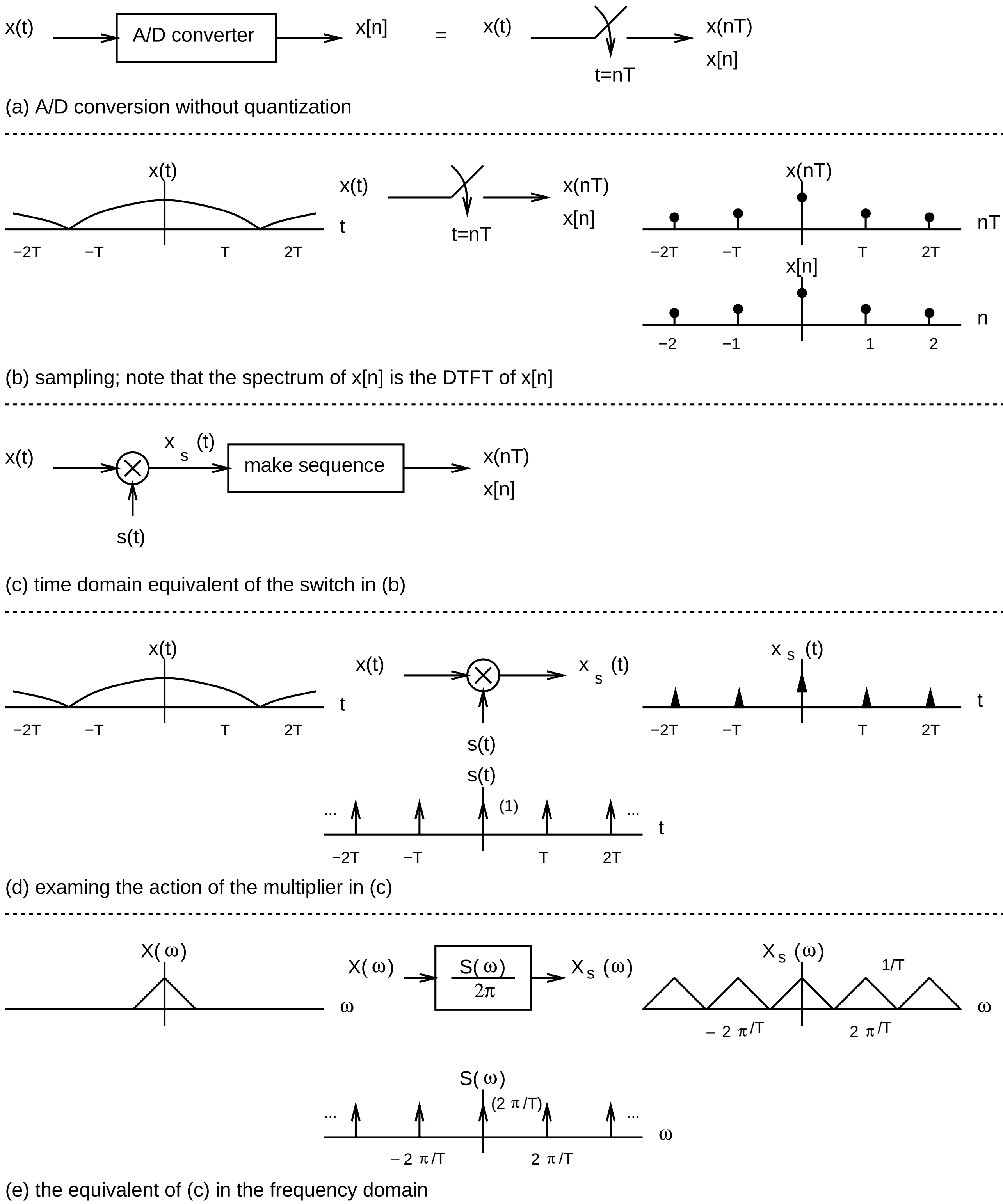

Figure 1: A/D conversion without quantization.

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)
\end{aligned}
$$

At this point in time, it is useful to note that the $x_s(t)$ is just the CT representation of the DT signal $x[n]$.

Now, let's take that entire setup in Figure 1(b) and consider what's going on in the frequency domain, as in Figure 1(c). Multiplication in the time domain becomes convolution in the frequency domain:

$$
\begin{aligned}
X_s(\omega) &= \frac{1}{2\pi} X(\omega) * S(\omega) \\
&= \frac{1}{2\pi} X(\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}) \\
&= X(\omega) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T})
\end{aligned}
$$

In English, this says the spectrum $X(\omega)$ of our original signal gets replicated at $\frac{2\pi}{T}$ intervals and scaled by $\frac{1}{T}$.

But wait! Two paragraphs previous, we found that a formula for $x_s(t)$. If we take the FT of that formula for $x_s(t)$, it should be equal to $\frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T})$. So let's take the FT of that formula for $x_s(t)$:

$$
\begin{aligned}
\mathcal{F}[x_s(t)] &= \int_{-\infty}^{\infty} [\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)] e^{-j\omega t} dt \\
&= \sum_{n=-\infty}^{\infty} x(nT) [\int_{-\infty}^{\infty} e^{-j\omega t}\delta(t-nT) dt] \\
&= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT} [\int_{-\infty}^{\infty} \delta(t-nT) dt] \\
&= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT} \\
&= X(e^{j\omega T})
\end{aligned}
$$

This is the DTFT of $x(nT)$. That means:

$$
X(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T})
$$

Is this much ado about nothing?[1] This tells us that if we want to find the DTFT, and we know it came from some CT signal, and we know what the spectrum $X(\omega)$ of that CT signal was, all we have to do is make copies of that spectrum at $\frac{2\pi}{T}$ intervals and scale by $\frac{1}{T}$. This should also square with what we know about the unnormalized DTFT, in that it's periodic with period $\frac{2\pi}{T}$.

For those who prefer the normalized version of things:

$$
\mathcal{F}[x_s(t)] = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}
$$

[1] If you think so, send me your CD player, your modem, your car, and anything else that has a DSP chip in it. It will supplement my generous TA salary of \$0.12. Yes, i got a raise from last semester– a whole one cent. We're really in the money here.

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$
$$= X(e^{j\Omega})$$

and this is equal to:

$$X(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T})$$
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\frac{\Omega}{T} - k\frac{2\pi}{T})$$

This means that you multiply all the values on the horizontal axis by the sampling period T, so that the DTFT is periodic with period 2π, which is the period of the normalized DTFT. You're stuck with the $\frac{1}{T}$ scaling factor on the vertical axis though.

3 Nyquist Sampling

Given that we know the spectrum of a sampled CT signal is copies of the spectrum $X(\omega)$ of the original CT signal at $\frac{2\pi}{T}$ intervals, under what conditions can we recover the original CT signal? When the copies don't overlap; if we assume that the CT signal is bandlimited such that $X(\omega) = 0$ for $|\omega| > \omega_{max}$, this happens when the maximum frequency ω_{max} is less than $\frac{2\pi}{T} - \omega_{max}$:

$$\omega_{max} < \frac{2\pi}{T} - \omega_{max}$$
$$\omega_{max} < \frac{\pi}{T}$$
$$2\frac{\omega_{max}}{2\pi} < \frac{1}{T}$$
$$2f_{max} < f_s$$

where the sampling rate f_s is the reciprocal of the sampling period T. If we then use an ideal reconstruction filter with spectrum $H_r(\omega) = T$ for $|\omega| < \omega_{max}$ and zero otherwise, we can recover the original CT signal.

To make sure that our spectrum does not overlap as we do the sampling operation, we could add an anti-aliasing filter into the front of everything, forcing the CT signal's spectrum to be nonzero for $|\omega| < \frac{\pi}{T}$. This would be good, unless we were interested in the parts of the spectrum that we end up throwing away.

This assumes that the signal has a spectrum that is nonzero for all $|\omega| < \omega_{max}$. If the signal is zero for some frequencies lower than ω_{max}, then we can sample at some frequency lower than $2f_{max}$. Consider a signal with 2 kHz bandwidth modulated up to 1MHz. We could sample at 2(1.002) MHz, but that would be a waste. If we sample at 4 kHz, we can use an ideal reconstruction filter to recover the original signal without having to demodulate.

Exercise Verify this.

4 D/A Conversion and the Zero-order Hold

A D/A converter takes a digital word and converts it into analog form using a zero-order hold, as in Figure 2(b). Unfortunately, the waveform that it generates looks something like a staircase. How do we interpret its spectrum? What can we do to reconstruct the desired waveform?

We can think of the D/A converter as taking some sequence, converting that sequence to its CT representation, and then convolving it with a shifted pulse of height 1, width T, and center at $t = \frac{T}{2}$. Let's call this shifted pulse $h_0(t)$. In Figure 2(c), we have taken a sampled cosine $x[n] = \cos \Omega_0 n$ as input into the D/A converter.

If we take everything into the frequency domain, as in Figure 2(d), we have the DTFT $X(e^{j\omega T})$ of the sequence $x[n]$, multiplied by the FT $H_0(\omega)$ of the pulse $h_0(t)$. We note that $X(e^{j\omega T})$ is periodic with period $\frac{2\pi}{T}$, and that:

$$H_0(\omega) = \frac{2\sin\omega\frac{T}{2}}{\omega} e^{-j\omega\frac{T}{2}}$$

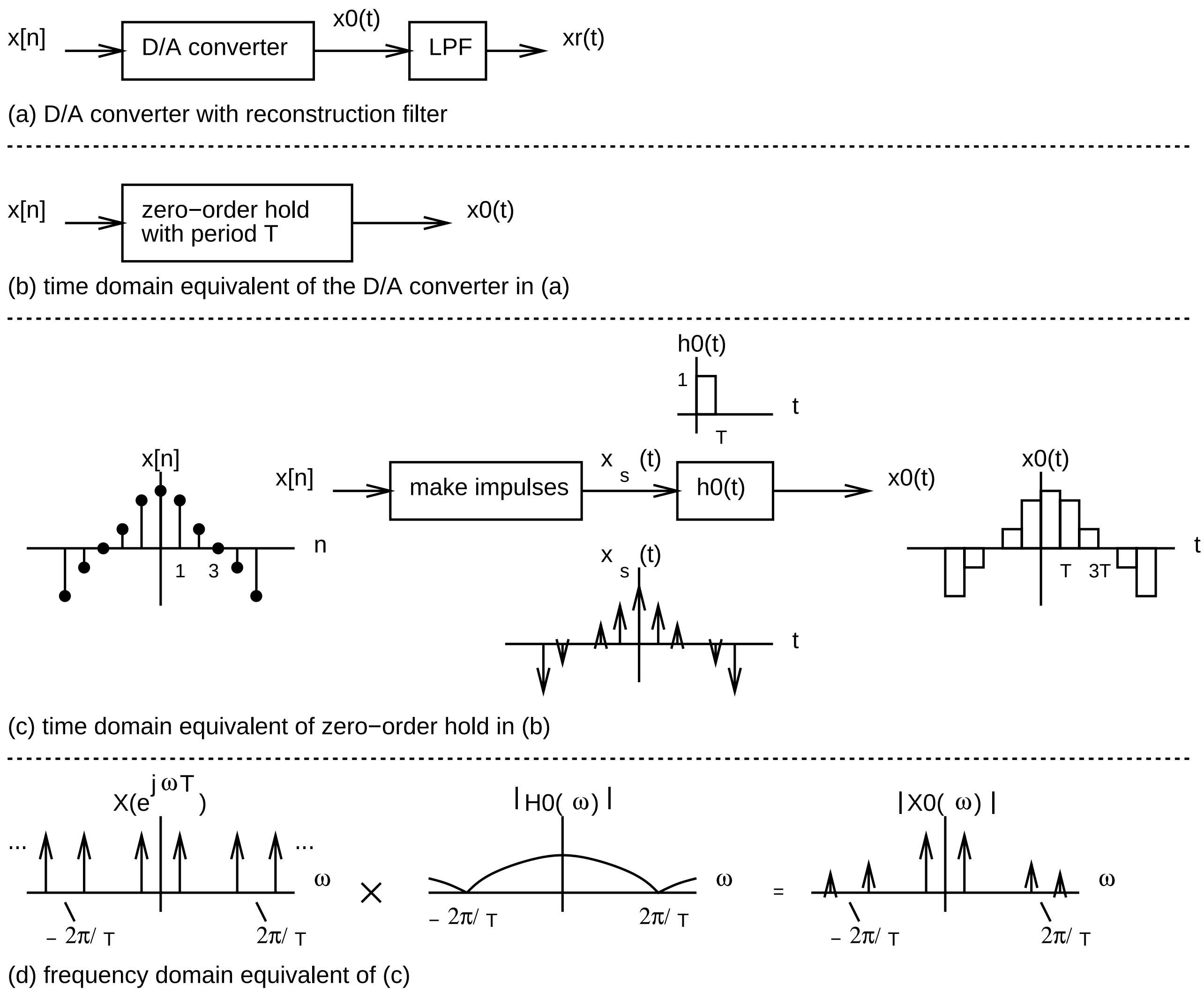

Figure 2: D/A conversion.

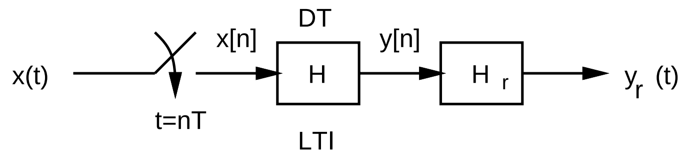

Figure 3: DT system with CT input, sampler, and reconstruction filter.

which is just a sinc with linear phase, zero crossings at $\frac{2\pi}{T}$. Conveniently, when the multiplication is performed, these zero crossings help to greatly attenuate copies of the spectrum at any other frequency other than baseband, as in Figure 2(d). This is good. Why?

Remember that this D/A converter is going to go in some system in order to reconstruct our sampled signal. If we pretend that we've sampled audio and put this staircase waveform that we get from the zero-order hold into a speaker, we're going to hear clicks and pops with period T, corresponding to the sharp edges of the steps in the staircase. Since edges are high frequencies, we can smooth out those edges using an analog low pass filter [this is the reconstruction filter]. In the frequency domain, those edges arise from the attenuated copies of the spectrum at multiples of $\frac{2\pi}{T}$ [other than at baseband]. So if we design an analog filter to pass only those frequencies with magnitude less than $\frac{\pi}{T}$, then we get our reconstructed signal. But since the sinc-shape from the zero-order hold attenuates everything other than the stuff at baseband, we can be a bit more sloppy designing the analog LPF.

Interested students [and those that want to get a jump on ee123] should consult Oppenheim and Schafer, sections 3.6 and 3.7. Note that they use Ω for frequency ω for normalized frequency, which is the reverse of the notation that we're using.[2]

5 DT Systems with CT Inputs and Outputs

How can we use a DT system, if we have CT inputs and CT outputs? One way is to sample the CT input to create a DT input, put that into the DT system, and then take the output of the DT system through an ideal reconstruction filter to get a CT output. The overall system can then be viewed as a CT system.

Consider the setup in Figure 3. The CT input $x(t)$ is sampled to form $x[n]$. DT system H takes $x[n]$ and creates $y[n]$, from which $y(t)$ is reconstructed. What's going on in the frequency domain?

From the second section of this set of notes, we know that the spectrum of $x[n]$ can be expressed in terms of the spectrum of $x(t)$:

$$X(e^{j\omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\frac{2\pi}{T})$$

The output of the $y[n]$ has a spectrum that can be expressed in terms of the spectrum of the input $x[n]$:

$$\begin{aligned} Y(e^{j\omega T}) &= H(e^{j\omega T})X(e^{j\omega T}) \\ &= \frac{1}{T}H(e^{j\omega T})\sum_{n=-\infty}^{\infty} X(\omega - n\frac{2\pi}{T}) \end{aligned}$$

However, we also know that $y[n]$ has a CT representation $y_s(t)$:

$$y_s(t) = \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT)$$

By section 2, the Fourier transform of $y_s(t)$, denoted $Y_s(\omega)$, is equal to the DTFT of $y[n]$:

$$\begin{aligned} Y_s(\omega) &= Y(e^{j\omega T}) \\ &= \frac{1}{T}H(e^{j\omega T})\sum_{n=-\infty}^{\infty} X(\omega - n\frac{2\pi}{T}) \end{aligned}$$

[2] i suppose they want you to build some more character.

If we put $y_s(t)$ into an ideal reconstruction filter with frequency response $H_r(\omega) = T\Pi(\frac{T\omega}{2\pi})$, the output of the filter, denoted $y_r(t)$, has the frequency response $Y_r(\omega)$:

$$\begin{aligned} Y_r(\omega) &= H_r(\omega)Y_s(\omega) \\ &= T\Pi(\frac{T\omega}{2\pi})\frac{1}{T}H(e^{j\omega T})\sum_{n=-\infty}^{\infty} X(\omega - n\frac{2\pi}{T}) \\ &= \Pi(\frac{T\omega}{2\pi})H(e^{j\omega T})\sum_{n=-\infty}^{\infty} X(\omega - n\frac{2\pi}{T}) \end{aligned}$$

If we know nothing about the frequency content of $x(t)$, this is as far as we can go. But in the case that the input is bandlimited [ie $X(\omega) = 0$ for all $|\omega| > \frac{\pi}{T}$], then we can further reduce $Y_r(\omega)$:

$$\begin{aligned} Y_r(\omega) &= \Pi(\frac{T\omega}{2\pi})H(e^{j\omega T})\sum_{n=-\infty}^{\infty} X(\omega - n\frac{2\pi}{T}) \\ &= H(e^{j\omega T})X(\omega) \end{aligned}$$

In other words, if we know that $x(t)$ is bandlimited to the range $-\frac{\pi}{T} < \omega < \frac{\pi}{T}$, then we can just design the DT system to have the desired frequency response in that range of frequencies, and ignore the sampling and reconstruction altogether.

Another way of looking at this is as follows: if the sampling occurs at a rate above the Nyquist rate [terminology for twice the maximum frequency in the signal], and we are using ideal reconstruction, you can completely forget about the sampling and reconstruction, and just design the DT system to have the desired frequency response in that range of frequencies.

Notice that it took us an entire semester to get to this point.