Notes 20 largely plagiarized by %khc

1 Warning

This set of notes covers discrete time. However, i probably won't be able to talk about everything here; instead i will highlight important properties or give random examples.[1] You are advised to consult your lecture notes and your textbook, since you are responsible for everything.

2 Discrete Time

We begin our discussion of discrete time by attempting to draw some analogies between continuous time (CT) and discrete time (DT).

In DT, our basic signal is $\delta[n]$. This function is nonzero at $n = 0$ only, as you would expect from a comparison with $\delta(t)$. The major difference is that $\delta[n]$ has a height of 1 at $n = 0$, whereas $\delta(t)$ has infinite height and area 1.

Our next most basic signal is $u[n]$, which can be constructed from $\delta[n]$ by taking the running sum from $k = -\infty$ to $k = n$:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Compare this to CT:

$$u(t) = \int_{\tau=-\infty}^{t} \delta(\tau) d\tau$$

This leads us to the conclusion that the local equivalent of integration in DT is the running sum. Differentiation in CT then becomes a finite difference, the difference between two or more successive samples.[2] Sometimes, CT differential equations are approximated by finite differences to convert them to DT difference equations.

One more thing before we move on: in DT, complex exponentials and sinusoids exhibiting a somewhat weird behavior. Consider two different complex exponentials in CT: $e^{j\omega_1 t}$ and $e^{j\omega_2 t}$. If ω_1 and ω_2 are not equal, then there is no way for the two complex exponentials to look the same as a function of time.

Now, consider the same two complex exponentials in DT. We'll do this by taking $t = nT$, effectively sampling the CT signal with sampling period T.[3] For the particular choice of $\omega_2 = \omega_1 + \frac{2\pi}{T}$:

$$\begin{aligned} e^{j\omega_2 nT} &= e^{j(\omega_1 + \frac{2\pi}{T})nT} \\ &= e^{j\omega_1 nT} e^{j\frac{2\pi}{T})nT} \\ &= e^{j\omega_1 nT} e^{j2\pi n} \\ &= e^{j\omega_1 nT} \end{aligned}$$

We can explicitly surpress the dependence on the sampling period T by defining a new variable $\Omega = \omega T$. We refer to ω as unnormalized frequency and to Ω as normalized frequency.

Anyway, in unnormalized or normalized frequency, we still end up with the same basic result— in unnormalized frequency, *you cannot tell the difference between a DT complex exponential at frequency ω_1 and another DT complex exponential* $\omega_2 = \omega_2 + \frac{2\pi}{T}$.

Exercise Show that in normalized frequency, the DT complex exponential Ω_2 and $\Omega_2 = \Omega_1 + 2\pi$ are the same value at integer values of n.

[1] Once again, randomness is good.

[2] The most basic finite difference is first order: $x[n] - x[n-1]$. More on this if you take math128a and math128b, numerical analysis parts I and II, which is highly recommended if you are going to end up coding a lot.

[3] More on sampling later. For now, let's think about sampling as closing a switch at multiples of time $t = nT$ and recording the resulting values.

3 DT Convolution

The DT equivalent of the sifting integral is:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This says that the entire sequence $x[n]$ can be obtained one sample at a time, by summing an infinite sequence of unit samples, appropriately scaled.

If we have a linear system H, and we take this $x[n]$ as input into our system, we end up with the output $y[n]$:

$$\begin{aligned} y[n] &= H[x[n]] \\ &= H[\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]] \\ &= \sum_{k=-\infty}^{\infty} x[k]H[\delta[n-k]] \end{aligned}$$

This is only from assuming linearity.

As with continuous time, we can define the impulse response $g[n,k] \triangleq H[\delta[n-k]]$ as a function of two variables n and k, where n can be interpreted as the current index, and k can be thought of as when the impulse occurred. Now, if the system is time-invariant, we don't really care too much *when* that impulse happens, only the difference between when that impulse happened and whatever index we're currently at. So $g[n,k]$ reduces from a function of two variables back down to a single variable: $g[n,k] = h[n-k]$.

A more elegant way of getting to the same result is to consider the definition of shift-invariance, which is the local equivalent of time invariance in DT.[4] Shift invariance says that if i shift a signal and put it into a system, the output is going to be the same as if i put the unshifted signal into the system first and shifted it later. In math, the delay and system operators commute:

$$H[D_k[x[n]]] = D_k[H[x[n]]]$$

Applying this definition of shift invariance, we note that:

$$\begin{aligned} H[\delta[n-k]] &= H[D_k[\delta[n]]] \\ &= D_k[H[\delta[n]]] \\ &= D_k[h[n]] \\ &= h[n-k] \end{aligned}$$

where we have defined $h[n] \triangleq H[\delta[n]]$, the impulse response. Note that i'm going to refer to shift invariance and time invariance interchangeably.[5]

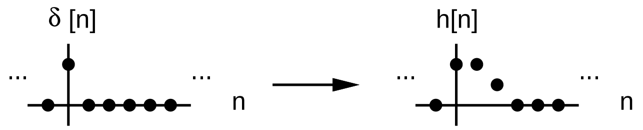
The final form of the convolution sum becomes:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

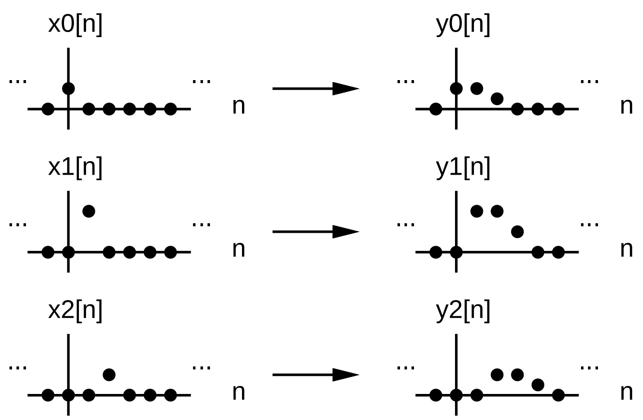
What are we actually doing here? Well, we can interpret our input as the sum of a bunch of scaled and shifted unit samples. If we know that our system is LTI and that it has some impulse response $h[n]$, then if we put in a δ function at $n-k$, the response of the system will be $h[n-k]$. Now, if we put in $\delta[n-k]$, scaled by $x[k]$, we will get $x[k]h[n-k]$. If we have a whole bunch of δ functions for various values of k, we can, by linearity, sum up the whole output as $\sum_{n=-\infty}^{\infty} x[k]h[n-k]$. This process is illustrated in Figure 1.

[4] The reason why the name is different is that the index n could represent some other variable such as position instead of time. For a concrete example, think of the position of a pixel in an image. Now, to warp your brain, consider a 3D image changing as a function of time...

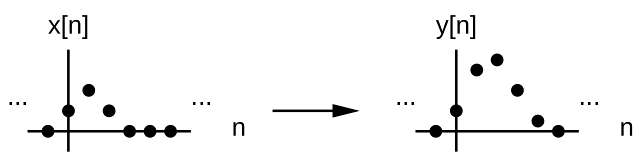
[5] Sorry, premature senility. Maybe i should take up skydiving now before i forget to pull the ripcord.



(a) an impulse and the corresponding impulse response



(b) scaling and shifting the impulse produces a scaled and shifted impulse response



(c) superposition of x0[n], x1[n], and x2[n] from (b) gives x[n];
likewise, superposition of y0[n], y1[n], and y2[n] gives y[n];
so if x[n] is input into our system with impulse response h[n],
the corresponding output is y[n]

Figure 1: DT convolution.

This convolution sum is somewhat easier to perform than the CT version. Please see OWY section 3.2 and ZTF section 8-4 for examples.[6]

Exercise Reconcile what you that about CT convolution with what you know about DT convolution.

Exercise i claim that you have already seen DT convolution as polynomial multiplication. For proof by example, multiply $x^2 + 2x + 1$ by $3x^2 + x + 2$ and then convolve $x[n] = \delta[n-2] + 2\delta[n-1] + \delta[n]$ and $y[n] = 3\delta[n-2] + \delta[n-1] + 2\delta[n]$. Compare your results.

4 DT Convolution Tricks

DT convolution is commutative, associative, and distributive over addition.

Exercise Prove this.

Convolution with an impulse gives back the original signal:

$$\begin{aligned}\delta[n] * x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\ &= x[n]\end{aligned}$$

where we have used the fact that $\delta[n-k]$ is nonzero only at $k = n$, and that at $n = k$ it takes on value 1.

Convolution with a unit step gives the running sum:

$$\begin{aligned}u[n] * x[n] &= \sum_{k=-\infty}^{\infty} x[k]u[n-k] \\ &= \sum_{k=-\infty}^{n} x[k]\end{aligned}$$

where we have used the fact that $u[n-k]$ is nonzero only for $n - k \geq 0$, or $k \leq n$.

Convolution with a shifted impulse shifts the original signal:

$$\begin{aligned}\delta[n-n_0] * x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-n_0-k] \\ &= x[n-n_0]\end{aligned}$$

where we have used the fact that $\delta[n-n_0-k]$ is nonzero only at $k = n - n_0$.

5 Eigenfunctions

The output $y[n]$ of a DT LTI system H is just the convolution of the input $x[n]$ with the impulse response $h[n]$. What if we let $x[n]$ be of the form $e^{j\Omega n}$, where ω is some real constant?

$$\begin{aligned}y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\Omega(n-k)}h[k] \\ &= e^{j\Omega n}\sum_{k=-\infty}^{\infty} e^{-j\Omega k}h[k] \\ &= e^{j\Omega n}H(e^{j\Omega})\end{aligned}$$

[6] OWY refers to the first edition of your textbook and ZTF to textbook used last semester. Be warned the DT sections in ZTF present the material in a sufficiently confusing fashion that i don't recommend those sections to you.

where $H(e^{j\Omega}) \triangleq \sum_{k=-\infty}^{\infty} e^{-j\Omega k} h[k]$. We will see later that $H(e^{j\Omega})$ is the discrete-time Fourier transform (DTFT) of $h[n]$.

Note also that $e^{j\Omega n}$ is an eigenfunction.

What if we let $x[n]$ be of the form z^n, where z is some complex constant?

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} z^{n-k} h[k] \\ &= z^n \sum_{k=-\infty}^{\infty} z^{-k} h[k] \\ &= z^n H(z) \end{aligned}$$

where $H(z) \triangleq \sum_{k=-\infty}^{\infty} z^{-k} h[k]$. We will see later that $H(z)$ is the Z transform of $h[n]$.

Note also that z^n is an eigenfunction.

6 Sinusoidal Steady State

What is we put $x[n] = \cos \Omega_0 n$ into a DT LTI system H?

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (\cos \Omega_0 n) * h[n] \\ &= \frac{1}{2}(e^{j\Omega_0 n} + e^{-j\Omega_0 n}) * h[n] \\ &= \frac{1}{2}(H(e^{j\Omega_0})e^{j\Omega_0 n} + H(e^{-jW_0})e^{-j\Omega_0 n}) \end{aligned}$$

If we know nothing about system H, this is all we can say about the output $y[n]$.

However, if we know that $h[n]$ is real, then we can say something about $H(e^{j\Omega})$. Consider $H^*(e^{j\Omega})$:

$$\begin{aligned} H^*(e^{j\Omega}) &= [\sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}]^* \\ &= \sum_{n=-\infty}^{\infty} h^*[n]e^{j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j(-\Omega)n} \\ &= H(e^{-j\Omega}) \end{aligned}$$

In other words, the DTFT $H(e^{j\Omega})$ is conjugate symmetric. If we rewrite this in polar:

$$\begin{aligned} H(e^{j\Omega}) &= |H(e^{j\Omega})|e^{j\angle H(e^{j\Omega})} \\ H(e^{-j\Omega}) &= |H(e^{-j\Omega})|e^{j\angle H(e^{-j\Omega})} \\ H^*(e^{j\Omega}) &= |H(e^{j\Omega})|e^{-j\angle H(e^{j\Omega})} \end{aligned}$$

The last two equations are equivalent, as derived above. So we can equate their magnitudes and phases. This lets us say that the magnitude is going to be even and the phase odd.

$$\begin{aligned} |H(e^{-j\Omega})| &= |H(e^{j\Omega})| \\ \angle H(e^{-j\Omega}) &= -\angle H(e^{j\Omega}) \end{aligned}$$

So what? From above, we had the output of a DT system for a sinusoidal input:

$$\begin{aligned} y[n] &= \frac{1}{2}(H(e^{j\Omega_0})e^{j\Omega_0 n} + H(e^{-jW_0})e^{-j\Omega_0 n}) \\ &= \frac{1}{2}[|H(e^{j\Omega_0})|e^{j\Omega_0 n + j\angle H(e^{j\Omega_0})} + |H(e^{-jW_0})|e^{-j\Omega_0 n + j\angle H(e^{-jW_0})}] \\ &= \frac{1}{2}[|H(e^{j\Omega_0})|e^{j\Omega_0 n + j\angle H(e^{j\Omega_0})} + |H(e^{jW_0})|e^{-j\Omega_0 n - j\angle H(e^{jW_0})}] \\ &= \frac{1}{2}|H(e^{j\Omega_0})|[e^{j\Omega_0 n + j\angle H(e^{j\Omega_0})} + e^{-j\Omega_0 n - j\angle H(e^{jW_0})}] \\ &= |H(e^{j\Omega_0})|\cos(\Omega_0 n + \angle H(e^{j\Omega_0})) \end{aligned}$$

In other words, a sinusoid input gives a sinusoid output, phase shifted by the phase of the frequency response and amplitude scaled by the magnitude response.

7 Representations

We begin our discussion of discrete time by noting that the sequence $x[n]$ and the sequence $x(nT)$ contain the same information. In fact, we can define $x_s(t)$ as the continuous time representation of $x(nT)$ such that:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

which is just an infinite sum of impulses; this also contains the same information as the plain old sequence $x[n]$ or $x(nT)$, except that it's a continous function, instead of a discrete one.

This should not be too surprising. We saw something similar to this when we represented the Fourier series coefficients a_k as a function of k [this is just a sequence or list of values, a function of a discrete variable], and as a function of ω [this is just an infinite sum of impulses, a function of a continuous variable].

8 Discrete-Time Fourier Transform (DTFT)

From the beginning of the semester, we saw the FT. Now, the FT transforms a continuous time signal $x(t)$ into a function of continous ω. What happens if we try this on our continous time representation of our sequence $x[n]$?

$$\begin{aligned} X_s(j\omega) &= \int_{-\infty}^{\infty} x_s(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} [\sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)]e^{-j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x(nT)[\int_{-\infty}^{\infty} e^{-j\omega t}\delta(t - nT)]dt \\ &= \sum_{n=-\infty}^{\infty} x(nT)[\int_{-\infty}^{\infty} e^{-j\omega nT}\delta(t - nT)]dt \\ &= \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}[\int_{-\infty}^{\infty} \delta(t - nT)]dt \\ &= \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} \end{aligned}$$

In continous time, we decomposed $x(t)$ into continous complex exponentials $e^{j\omega t}$. In discrete time, we have decomposed $x(t)$ into discrete complex exponentials.

If we replace $x(nT)$ by $x[n]$, this last quantity is the DTFT:

$$X(e^{j\omega t}) \stackrel{\Delta}{=} \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT}$$

The notation $X(e^{j\omega t})$ denotes the DTFT. This notation is more widely used than the one in the textbook.

Note that the discrete complex exponential $e^{j\omega nT}$ is periodic with period $\frac{2\pi}{T}$:

$$\begin{aligned} e^{j(\omega+\frac{2\pi}{T})nT} &= e^{j\omega nT}e^{2\pi n} \\ &= e^{j\omega nT} \end{aligned}$$

This means that $X(e^{j\omega T})$ will be periodic with period $\frac{2\pi}{T}$. Now a slightly weird thing happens– low frequencies show up at multiples of $\frac{2\pi}{T}$ and high frequencies end up at multiples of $\frac{\pi}{T}$.

To further supress the time dependence, the notation $\Omega = \omega T$ is introduced, in which case the transform $X(e^{j\Omega})$ is periodic with period 2π; low frequencies then occur at multiples of 2π, and high frequencies at multiples of π.

Exercise Given that $\Omega = \omega T$, convince yourself that the units of Ω are radians per cycle.

A slightly more intuitive reason for why the DTFT is periodic: since you can't tell the difference between $e^{j\omega nT}$ and $e^{j(\omega+k\frac{2\pi}{T})nT}$, you shouldn't be able to tell what frequency at which it appears in the frequency domain either.

There is another interpretation of the DTFT, arising from the fact that you can think of the time signal as the sampled version of some continuous time signal $x(t)$. To sample in time, you multiply in time by an impulse train. This means that you convolve in frequency with an impulse train, which makes your spectrum periodic. T can then be interpreted as the sampling period. More on this in the last few lectures.

Note that multiplying in time by an impulse train is a mathematical trick that we use in order to derive the DTFT. What we can actually do is sample our continous time $x(t)$ at periodic intervals $t = nT$ to obtain $x(nT)$. Repeat: multiplying in time by an impulse train is a math trick.

The DTFT has a number of properties which are summarized on pages 335-336 of OWY; common DTFT transform pairs are given on OWY pages 338-339.

Personally, i prefer to work with the Z transform, and then utilize the relationship between the Z transform and the DTFT to find a given DTFT. On the other hand, if you know that a certain signal $x[n]$ came from sampling $x(t)$ and you know $X(\omega)$, then you can quickly find the DTFT by making copies of $X(\omega)$, separated by $\frac{2\pi}{T}$ and scaled by $\frac{1}{T}$. More on this in the next set of notes.

9 Fourier Series, Revisited

Assume that $x(t)$ is periodic with period $T = \frac{2\pi}{\omega_0}$. Also, assume that $x(t)$ is sufficiently well-behaved such that we can represent it by a Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

The Fourier series coefficients X_n can be found by multiplying both sides by $e^{-jm\omega_0 t}$ and integrating over one period T:

$$\begin{aligned} \int_T x(t)e^{-jm\omega_0 t}\,dt &= \int_T \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt \\ &= \sum_{n=-\infty}^{\infty} X_n \int_T e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt \end{aligned}$$

We recall the orthogonality relationship for complex exponentials:

$$\int_T e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt = \begin{cases} T & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

This lets us reduce the equation above to:

$$\begin{aligned}\int_T x(t)e^{-jm\omega_0 t}\,dt &= \sum_{n=-\infty}^{\infty} X_n \int_T e^{jn\omega_0 t}e^{-jm\omega_0 t}\,dt \\ &= TX_m \\ X_m &= \frac{1}{T}\int_T x(t)e^{-jm\omega_0 t}\,dt\end{aligned}$$

This last equation is the Fourier series analysis integral.

If we take the Fourier transform of $x(t)$:

$$\begin{aligned}X(\omega) &= \mathcal{F}[x(t)] \\ &= \mathcal{F}[\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}] \\ &= \sum_{n=-\infty}^{\infty} X_n \mathcal{F}[e^{jn\omega_0 t}] \\ &= 2\pi \sum_{n=-\infty}^{\infty} X_n \delta(\omega - n\omega_0)\end{aligned}$$

where the X_n can be found by the Fourier series analysis integral above.

Note that X_n is just a sequence of values, indexed on n. The time signal to which it corresponds in the time domain is periodic in t.

10 DTFT, Revisited

Assume that $X(e^{j\omega T})$ is periodic with period $\omega_0 = \frac{2\pi}{T}$, where T is the sampling period. We further assume that $X(e^{j\omega T})$ is sufficiently well-behaved such that we can represent it by a Fourier series:

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega T}$$

Note that we choose to use a negative sign, since we're in the frequency domain.

The Fourier series coefficient $x[n]$ can be found by multiplying both sides by $e^{jm\omega T}$ and integrating over one period ω_0:

$$\begin{aligned}\int_{\omega_0} X(e^{j\omega T})e^{jm\omega T}\,d\omega &= \int_{\omega_0} \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega T}e^{jm\omega T}\,d\omega \\ &= \sum_{n=-\infty}^{\infty} x[n]\int_{\omega_0} e^{-jn\omega T}e^{jm\omega T}\,d\omega\end{aligned}$$

We recall the orthogonality relationship for complex exponentials:

$$\int_{\omega_0} e^{-jn\omega T}e^{jm\omega T}\,d\omega = \begin{cases}\omega_0 & \text{if } n = m \\ 0 & \text{otherwise}\end{cases}$$

This lets us reduce the equation above to:

$$\begin{aligned}\int_{\omega_0} X(e^{j\omega T})e^{jm\omega T}\,d\omega &= \sum_{n=-\infty}^{\infty} x[n]\int_{\omega_0} e^{-jn\omega T}e^{jm\omega T}\,d\omega \\ &= \omega_0 x[m]\end{aligned}$$

$$\begin{aligned} x[m] &= \frac{1}{\omega_0} \int_{\omega_0} X(e^{j\omega T}) e^{jm\omega T} d\omega \\ &= \frac{T}{2\pi} \int_{\frac{2\pi}{T}} X(e^{j\omega T}) e^{jm\omega T} d\omega \end{aligned}$$

This is the inverse DTFT.

Taking the inverse Fourier transform of $X(e^{j\omega T})$:

$$\begin{aligned} x(t) &= \mathcal{F}^{-1}[X(e^{j\omega T})] \\ &= \mathcal{F}^{-1}[\sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega T}] \\ &= \sum_{n=-\infty}^{\infty} x[n] \mathcal{F}^{-1}[e^{-jn\omega T}] \\ &= \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT) \end{aligned}$$

This is the DTFT.

Note that $x[n]$ is just a sequence of values, indexed on n. Its Fourier transform is periodic in ω.

$x(t)$ is just the continuous time representation of the discrete time signal $x[n]$ as discussed above.

Now we have the entire DTFT pair:

$$\begin{aligned} X(e^{j\omega T}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega T_0} \\ x[n] &= \frac{T}{2\pi} \int_{\frac{2\pi}{T}} X(e^{j\omega T}) e^{jn\omega T_0} d\omega \end{aligned}$$

In normalized frequency, the DTFT pair becomes:

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \\ x[n] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{jn\Omega} d\omega \end{aligned}$$

In other words, *periodic in one domain implies discrete in the other*. Here, periodic in frequency [$X(e^{j\Omega})$] implies discrete in time [the $x[n]$], indexed on n. From the previous section, periodic in time [$x(t)$] implies discrete in frequency [the X_n], indexed on n.

Exercise Doesn't this look familiar? i lifted it from ps7 solutions. Take another look and make sure that you can see the duality relationship between the FS and the DTFT.

11 Z-Transform (ZT)

As with the DTFT, if we start out in continuous time and then take the Laplace transform of $x_s(t)$:

$$\begin{aligned} X_s(s) &= \int_{-\infty}^{\infty} x_s(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} [\sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)] e^{-st} dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) [\int_{-\infty}^{\infty} e^{-st} \delta(t - nT)] dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} x(nT)[\int_{-\infty}^{\infty} e^{-snT}\delta(t-nT)]dt \\
&= \sum_{n=-\infty}^{\infty} x(nT)e^{-snT}[\int_{-\infty}^{\infty} \delta(t-nT)]dt \\
&= \sum_{n=-\infty}^{\infty} x(nT)e^{-snT} \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\end{aligned}
$$

where we have made the change of notation $x(nT) = x[n]$ and $z = e^{sT}$.

This development is similar to that of the DTFT. So why do we bother studying the Z transform? For the same reason as why we study the Laplace transform— there are some signals for which the DTFT does not exist.

What is $z = e^{sT}$? It's a conformal mapping between the s-plane and the z-plane, taking vertical lines in the s-plane and making them into circles in the z-plane [consider $s = \sigma + j\omega$ and note that $e^{\sigma T}$ is the radius of a circle and $e^{j\omega T}$ is the angle]. In particular, the $j\omega$ axis ends up as the unit circle, the left half plane ends up inside the unit circle, and the right half plane ends up outside. This also suggests that, for a system to be stable, its Z transform should have poles inside the unit circle.

Exercise Verify the contents of this paragraph.

For some values of z, the sum will not converge. Those values for which the sum does converge comprise the region of convergence (ROC) [compare to the ROC of the Laplace transform]. The ROC is a circle centered on the origin, with its radius the distance from the origin to the outermost pole. Because we consider only the unilateral Z transform, the ROCs all go outwards.

The properties are summarized on OWY page 654, and are scattered throughout section 8-3 of ZTF; common transforms are available on OWY page 655, ZTF page 378.

12 Relationships

The relationships between the five transforms that you have studied and the DTFS/DFS/DFT that you will see again in ee123 are summarized in the following table:

LT:$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	$\overset{z=e^{sT}}{\leftrightarrow}$	ZT:$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$
$s = j\omega \downarrow$		$z = e^{j\omega T} \downarrow$
FT:$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	$\overset{discretize:t=nT}{\rightarrow}$	DTFT:$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT}$
$periodicize \downarrow$		$periodicize \downarrow$
FS:$a_k = \int_T x(t)e^{-jk\omega_0 t}dt$	$\overset{discretize:t=nT}{\rightarrow}$	DFS/DFT

Given that you have the Laplace transform of some time signal $x(t)$, if you evaluate the transform on the $j\omega$ axis, you end up with the Fourier transform. If you make the time signal periodic, you end up with a discrete spectrum [Fourier series].

On the other hand, we could start with a continous time signal $x(t)$ and make it discrete. If we take its Laplace transform and utilize the conformal mapping $z = e^{sT}$, where T is the sampling period, we end up with the Z transform. Evaluating the Z tranform on the unit circle gives you the the DTFT.

To get from the continuous time transforms to the discrete time transforms, we can *think* of taking our CT $x(t)$ and multiplying it by an impulse train to get $x(nT)$. This is just a math trick though— since we only care about the $x(nT)$, we can get this by sampling $x(t)$ at intervals $t = nT$ [perhaps by using a sample-and-hold, followed by an A/D converter].

Exercise Follow the links between the six transforms and try to get everything straight in your mind. There's a lot of material summarized in that one table, so make sure that it makes sense. Drop by during office hours if not.