Notes 11 largely plagiarized by %khc

1 Echo Compensation

[adapted from fall94, ps5] A room with multiple echoes can be modelled, for fixed source and receiver locations, as an LTI system with impulse response $h(t) = \delta(t) + \alpha\delta(t - T_0) + \alpha^2\delta(t - 2T_0) + \ldots$, where $|\alpha| < 1$ and T_0 is the delay time.
(a) Determine the magnitude and phase responses $H(\omega)$ and $\angle H(\omega)$.
(b) Design an inverse filter $G(\omega)$ such that $G(\omega)H(\omega) = 1$.
(c) Determine the impulse response $g(t)$ of the inverse filter.
We first need to find $H(\omega)$. Using the linearity and time shift properties, as well as the FT of $\delta(t)$, we have:

$$\begin{aligned}
H(\omega) &= 1 + \alpha e^{-j\omega T_0} + \alpha^2 e^{-j\omega 2T_0} + \ldots \\
&= \frac{1}{1 - \alpha e^{-j\omega T_0}} \\
|H(\omega)| &= \frac{1}{|1 - \alpha(\cos\omega T_0 - j\sin\omega T_0)|} \\
&= \frac{1}{|1 - \alpha\cos\omega T_0 + j\alpha\sin\omega T_0|} \\
&= \frac{1}{\sqrt{(1 - \alpha\cos\omega T_0)^2 + \alpha^2\sin^2\omega T_0}} \\
&= \frac{1}{\sqrt{1 - 2\alpha\cos\omega T_0 + \alpha^2}} \\
\angle H(\omega) &= \frac{\angle 1}{\angle(1 - \alpha(\cos\omega T_0 - j\sin\omega T_0))} \\
&= -\arctan\frac{\alpha\sin\omega T_0}{1 - \alpha\cos\omega T_0}
\end{aligned}$$

Why would we be interested in $G(\omega)$ in the first place? Well, what we have is an LTI system H that takes $x(t)$ and produces some $y(t)$. We want to make an LTI system G that takes $y(t)$ and produces an output $z(t)$ that is equivalent to $x(t)$. In other words, we want the composite system GH to take input $x(t)$ and give us output $x(t)$ again. The system which does this is the identity system with impulse response $\delta(t)$; the composite system has [by problem set 2, problem 5] impulse response $g(t) * h(t)$. In the frequency domain, we have the condition $G(\omega)H(\omega) = 1$.

So, the inverse system G has a frequency response $G(\omega) = \frac{1}{H(\omega)}$, or:

$$G(\omega) = 1 - \alpha e^{-j\omega T_0}$$

So by table lookup, $g(t)$ is then

$$g(t) = \delta(t) - \alpha\delta(t - T_0)$$

How do we interpret this impulse response? Well, $\delta(t)$ is the impulse response of a wire (the indentity system: the output of the system is equivalent to the input) and $\delta(t - T_0)$ is the impulse response of a delay by T_0. So we are taking the signal with echoes and subtracting an appropriately scaled and delayed version of that signal to recover our original signal.

Exercise Sketch the magnitude and phase responses. Check the math. Verify that $g(t) * h(t) = \delta(t)$. See if the inverse filter makes any sense.

2 More Problems

[mutated from fall94, midterm I]

A A system with input $x(t)$ and output $y(t)$ is described by $2\ddot{y}+4\dot{y}+3y=5\dot{x}+7x$. What is its transfer function?
Use the FT differentiation property: $\dot{x}(t)\leftrightarrow j\omega X(\omega)$.

$$\begin{aligned}(-2\omega^2+4j\omega+3)Y(\omega) &= (5j\omega+7)X(\omega)\\ \frac{Y(\omega)}{X(\omega)} &= \frac{5j\omega+7}{-2\omega^2+4j\omega+3}\end{aligned}$$

B An LTI system has transfer function $\frac{Y(\omega)}{X(\omega)}=j\omega+1+\frac{1}{j\omega}$. What is the impulse response of the system?
Heavy use of our FT table reveals that $\dot{\delta}(t)\leftrightarrow j\omega$, $\delta(t)\leftrightarrow 1$, and sgn $(t)\leftrightarrow\frac{2}{j\omega}$. So $h(t)=\dot{\delta}(t)+\delta(t)+\frac{1}{2}\text{sgn }(t)$.

C An LTI system has impulse reponse $h(t)=e^{-t}u(t)$. What is the output $y(t)$ if $x(t)=e^{j\omega_0 t}$?
$e^{j\omega t}$ is an eigenfunction for all LTI systems. So $y(t)=H(\omega_0)e^{j\omega_0 t}$. But $H(\omega)=\frac{1}{j\omega+1}$. Therefore, $y(t)=\frac{1}{j\omega_0+1}e^{j\omega_0 t}$.

D An LTI system has step response $g(t)=(1+e^{-t})u(t)$. What is the output $y(t)$ if the input is $x(t)=\cos\omega_0 t$?
We know that the impulse response $h(t)$ is the derivative of the step response. So if we use the property that $\dot{x}(t)\leftrightarrow j\omega X(\omega)$, we can find the frequency response.

$$\begin{aligned}H(\omega) &= \mathcal{F}\{\frac{d}{dt}[u(t)+e^{-t}u(t)]\}\\ &= j\omega[\frac{1}{j\omega}+\pi\delta(\omega)+\frac{1}{j\omega+1}]\\ &= 1+j\omega\pi\delta(\omega)+\frac{j\omega}{j\omega+1}\\ &= 1+\frac{j\omega}{j\omega+1}\end{aligned}$$

Then, since $h(t)$ is real, the output $y(t)$ is then $|H(\omega_0)|\cos(\omega_0+\angle H(\omega_0))$. So $y(t)=(1+\frac{\omega_0^2}{\omega_0^2+1})\cos(\omega_0 t+\angle H(\omega_0))$, where $\angle H(\omega_0)=\arctan 2\omega_0-\arctan\omega_0$.

3 Quick Questions

[adapted from fall94, midterm I; sketches in Figure 1]

A An LTI system has a unit step output for a unit ramp input. What would the time output of the system be for a square wave input?
If $r(t)*h(t)=u(t)$, then $h(t)=\dot{\delta(t)}$, the impulse response of a differentiator. So a square wave input is going to give you the derivative of that square wave as an output. If the square wave is ±1 peak-to-peak, then the derivative is going to have impulse alternately pointed up and down with area of ±2 [compare to the derivative of $u(t)$ to explain the area of 2].

B An LTI system has impulse response $h(t)=\cos\frac{2\pi}{T}t\Pi(\frac{t}{T})$. What is the step response?
The step response is the running integral of the impulse response. So the output is going to be one cycle of a sine, specifically $\sin\frac{2\pi}{T}\Pi(\frac{t}{T})$ [a problem similar to this was on ps4].

C A 100Hz square wave is passed through an LTI system with frequency response $H(\omega)$. What is the output spectrum of the filter?
A similar problem was solved in notes07. Determine the frequencies that pass through the filter by evaluating $H(\omega)$ and values of $k\omega_0$. In this case, the only frequency that makes it through the filter is the fundamental at 200π.

D An LTI system has impulse response $h(t)$ and input $x(t)$. What is output $y(t)$ of the system? What is the output of the system at $t=5$?

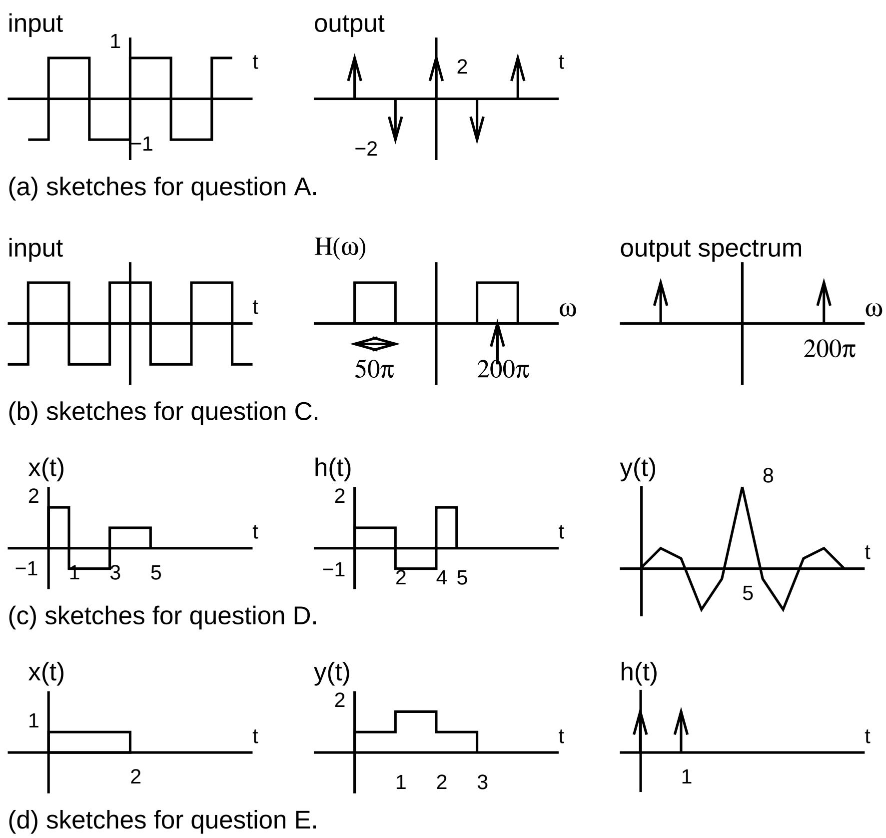

Figure 1: Sketches for the quick questions.

A straight flip and drag problem. Note that the output should be symmetric. Also note the large response at $t = 5$. [By the way, how do you quickly find the response at $t = 5$? Draw $h(5 - \tau)$ and $x(\tau)$ on the same set of axes. Find the area under their product.] This is the concept behind matched filtering. If you are looking for a particular waveform in some signal $s(t)$, you can convolve that signal $s(t)$ with the time-reversed version of that waveform. When you get a large response, you'll have pinpointed when that waveform occurs in $s(t)$.

E An LTI and causal system has input $x(t)$ and output $y(t)$. What is the output for an impulse input? See the sketches in Figure 1. [A very similar problem was on ps4.]

F What is the spectrum of $\cos(100t)[u(t + 0.001) - u(t - 0.001)]$?
We could crank through the FTs, using the property that multiplication in the time domain is equivalent to convolution in the frequency domain. Or we could note that since the pulse is very narrow and the frequency of the cosine is low in comparison, multiplying the cosine by the pulse gives you back something that looks very much like a pulse. So the spectrum is going to look quite a lot like a sinc. Yes, i'm too lazy to draw it.