Notes 08 largely plagiarized by %khc

1 Fourier Transform and Inverse Fourier Transform

We've seen the Fourier transform and its inverse a billion times.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \end{aligned}$$

But does $\mathcal{F}^{-1}\mathcal{F}[x(t)] = x(t)$? Let's check it out:

$$\begin{aligned} \mathcal{F}^{-1}\mathcal{F}[x(t)] &= \mathcal{F}^{-1}[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'] e^{j\omega t} d\omega \end{aligned}$$

At this point in time, we use the time-honored method of interchanging the integrals without bothering to justify this step.

$$\mathcal{F}^{-1}\mathcal{F}[x(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t') [\int_{-\infty}^{\infty} e^{j\omega(t-t')} d\omega] dt'$$

Let's take some time to check out that inner integral.

$$\begin{aligned} \int_{-\infty}^{\infty} e^{j\omega(t-t')} d\omega &= \lim_{W\to\infty} \int_{-W}^{W} e^{j\omega(t-t')} d\omega \\ &= \lim_{W\to\infty} \frac{1}{j(t-t')} e^{j\omega(t-t')}|_{-W}^{W} \\ &= \lim_{W\to\infty} \frac{1}{j(t-t')} [e^{jW(t-t')} - e^{-jW(t-t')}] \\ &= \lim_{W\to\infty} \frac{2\sin W(t-t')}{(t-t')} \end{aligned}$$

Warning: handwaving follows. As $W \to \infty$, the frequency of the sine goes through the ceiling. This means that if we move $t - t'$ ϵ units left or right, the sine is going to give us a drastically different value, but its average value is going to be effectively zero. So this integral will be equal to zero for all values of $t - t'$ not equal to zero.

Only at $t - t' = 0$ do we have a nonzero value. Since the area of the sinc does not depend on frequency $[\int_{-\infty}^{\infty} \frac{\sin Wt}{t} dt = \pi]$, all the area ends up right under an impulse centered at $t - t' = 0$. So

$$\int_{-\infty}^{\infty} e^{j\omega(t-t')} d\omega = 2\pi\delta(t-t')$$

We then have:

$$\begin{aligned} \mathcal{F}^{-1}\mathcal{F}[x(t)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t') [\int_{-\infty}^{\infty} e^{j\omega(t-t')} d\omega] dt' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t') 2\pi\delta(t-t') dt' \\ &= \int_{-\infty}^{\infty} x(t')\delta(t-t') dt' \\ &= x(t) \end{aligned}$$

using the sifting integral.[1]

[1] This material was lifted from Prof. Fearing's lecture notes from fall 1994.