Notes 04 largely plagiarized by %khc

1 Convolution Recap

Some tricks:

- $x(t) * \delta(t) = x(t)$
- $x(t) * \delta(t-t_0) = x(t-t_0)$
- $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- $x(t) * \dot{\delta}(t) = \dot{x}(t)$

This then tells us that an integrator has impulse response $h(t) = u(t)$, and that a differentiator has impulse response $h(t) = \dot{\delta}(t)$.

Convolution is associative and commutative. Convolution also distributes over addition.

Exercise Prove it.

Convolution with $h(t)$ is both linear and time invariant.

$$\begin{aligned}
[\alpha x_1(t) + \beta x_2(t)] * h(t) &= [\alpha x_1(t)] * h(t) + [\beta x_2(t)] * h(t) \\
&= \alpha[x_1(t) * h(t)] + \beta[x_2(t) * h(t)] \\
&= \alpha y_1(t) + \beta y_2(t) \\
x(t-t_0) * h(t) &= [x(t) * \delta(t-t_0)] * h(t) \\
&= [x(t) * h(t)] * \delta(t-t_0) \\
&= y(t) * \delta(t-t_0) \\
&= y(t-t_0)
\end{aligned}$$

This should not surprise you. If $y(t)$ is the output of some LTI system with input $x(t)$, then $y(t) = x(t) * h(t)$.

2 More Convolution Fun

We now dig into the old ee120 archives and drag out some old problems to illustrate convolution. [Sketches of the output for each of the parts are available. Please see Figure 1.]

A. An LTI system has impulse response $h(t) = e^{-t/2}u(t)$, input $x(t) = x_1(t)\sum_{n=-\infty}^{\infty}\delta(t-n)$, and output $y(t)$. If $x_1(t) = u(t+0.01) - u(t-3.01)$, determine the output.

First determine $x(t)$. Notice that $x(t)$ is just an impulse train multiplied by a pulse from $t = -0.01$ to $t = 3.01$. This just picks out the four impulses at $t = 0$, $t = 1$, $t = 2$, and $t = 3$. Then use superposition. Convolution with shifted impulses gives four shifted copies of $h(t)$. So $y(t) = e^{-t/2}u(t) + e^{-(t-1)/2}u(t-1) + e^{-(t-2)/2}u(t-2) + e^{-(t-3)/2}u(t-3)$.

B. $x(t) = e^{-2t}u(t)$. $h(t) = 2\delta(t-1) + \delta(t+1)$.

Same tricks as above. $y(t) = 2e^{-2(t-1)}u(t-1) + e^{-2(t+1)}u(t+1)$.

C. $x(t) = \sum_{n=-\infty}^{\infty}\delta(t-0.01n)$. $h(t) = \Pi(200t)$.

$x(t)$ is a train of impulses, each separated from the next by $\frac{1}{100}$. $h(t)$ is a pulse of height 1 from $t = -\frac{1}{400}$ to $t = \frac{1}{400}$. Convolving $h(t)$ with an impulse centered at $t = t_0$ gives $h(t-t_0)$. Since there are infinitely many shifted impulses, there are infinitely many copies of $h(t)$, whose centers are separated from that of their nearest neighbors by $\frac{1}{100}$. So $y(t) = \sum_{n=-\infty}^{\infty}\Pi[200(t-\frac{n}{100})]$.

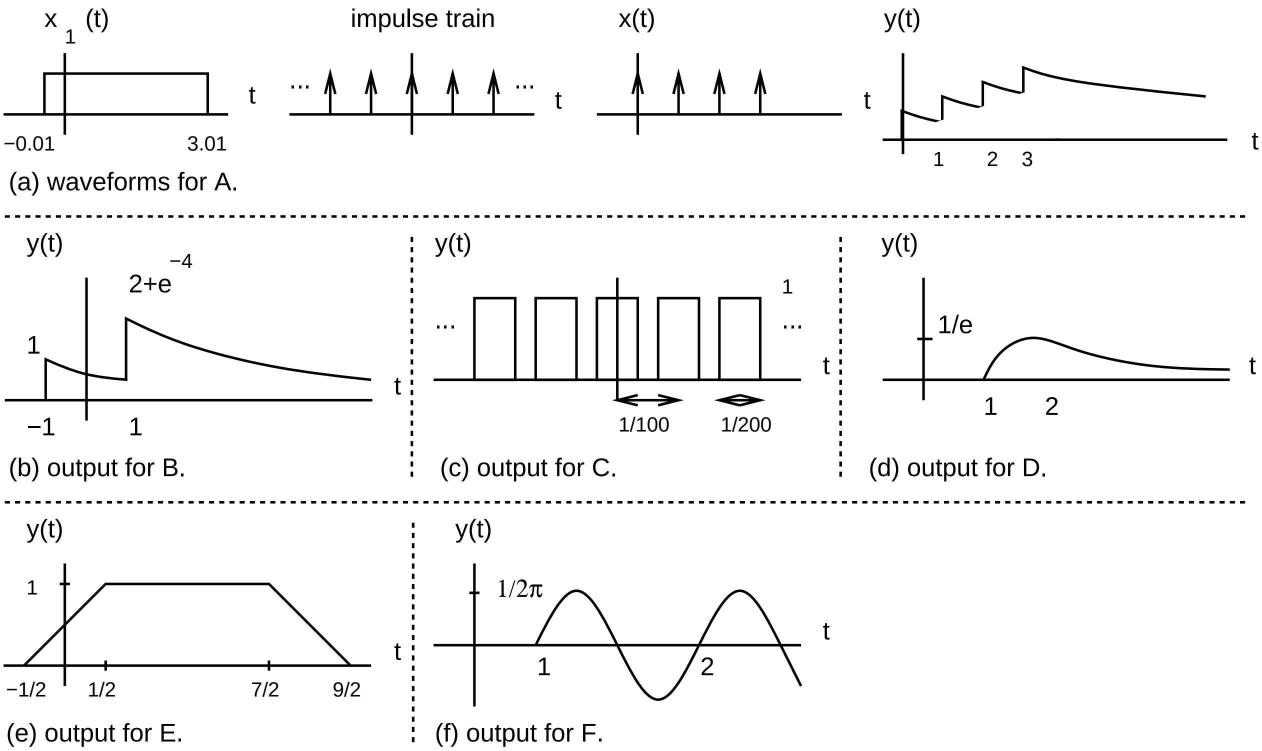


Figure 1: Sketches of the example problems. Knowing what the output looks like is definitely as good as having a mathematical formula.

D. $x(t) = e^{-t}u(t)$. $h(t) = r(t-1)\Pi(t-\frac{3}{2})$.

Let's choose $x(t)$ to flip and shift. There are three places where the integral is different.

For $t < 1$, there is no overlap, so $y(t) = 0$.

For $1 < t < 2$,

$$\begin{aligned} y(t) &= \int_1^t e^{-(t-\tau)}(\tau-1)d\tau \\ &= e^{-t}\int_1^t e^{\tau}(\tau-1)d\tau \\ &= e^{-t}(\tau e^{\tau} - 2e^{\tau})|_1^t \\ &= t - 2 + e^{1-t} \end{aligned}$$

For $t > 2$,

$$\begin{aligned} y(t) &= \int_1^2 e^{-(t-\tau)}(\tau-1)d\tau \\ &= e^{-t}\int_1^2 e^{\tau}(\tau-1)d\tau \\ &= e^{-t}(\tau e^{\tau} - 2e^{\tau})|_1^2 \\ &= e^{-(t-1)} \end{aligned}$$

E. $x(t) = \Pi(t)$. $h(t) = \Pi(\frac{t-2}{4})$.

You can do flip and shift, make the pulses into sums of unit steps and convolve, or just use your intuition.

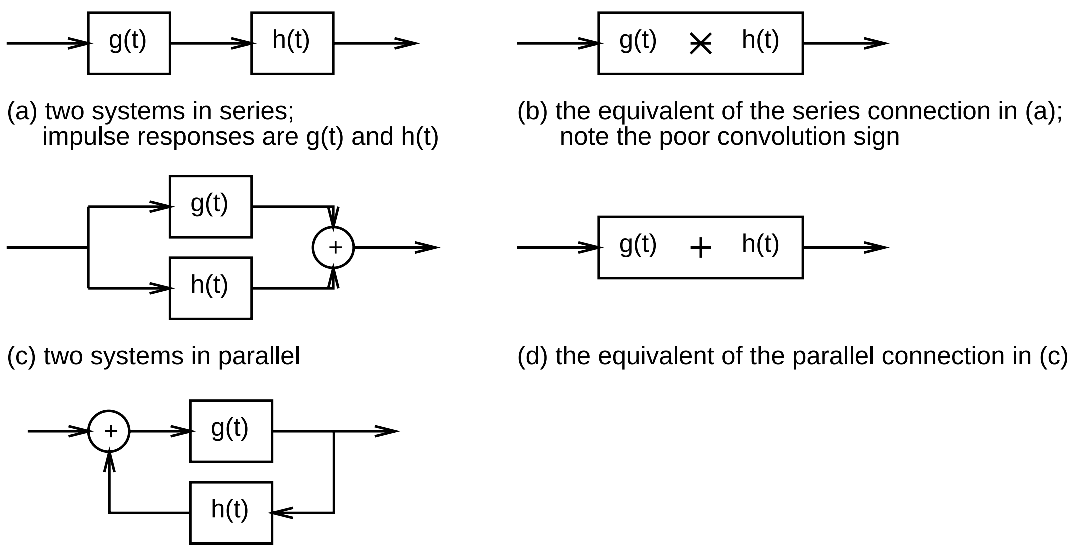
$$y(t) = \begin{cases} 0 & \text{for } t < -\frac{1}{2} \\ t + \frac{1}{2} & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < t < \frac{7}{2} \\ \frac{9}{2} - t & \text{for } \frac{7}{2} < t < \frac{9}{2} \end{cases}$$

F. $x(t) = u(t-1)$. $h(t) = \cos(2\pi t)u(t)$.

Convolve $u(t)$ with $h(t)$. This gives $y(t) = \frac{1}{2\pi}\sin(2\pi t)u(t)$. Now, since the unit step is actually delayed by 1, delay $y(t)$ by 1. So $y(t) = \frac{1}{2\pi}\sin[2\pi(t-1)]u(t-1)$.

3 System Interconnections

There are three major ways of putting systems together. Check out Figure 2 and see if the equivalents make sense to you [two of them already should, since you saw them on ps2, problem 6]. The reason why we haven't fully talked about the feedback configuration is that the analysis becomes much easier when we hit the Fourier and Laplace transforms. Wait 1.5 months.



(a) two systems in series; impulse responses are g(t) and h(t)

(b) the equivalent of the series connection in (a); note the poor convolution sign

(c) two systems in parallel

(d) the equivalent of the parallel connection in (c)

(e) two systems in feedback connection

(g) unfortunately, we're not prepared to talk about the equivalent of the feedback connection yet in a simple fashion, but we will...

Figure 2: The three major ways we have of composing systems.

4 Eigenfunctions

Consider an LTI system with input $x(t)$, impulse response $h(t)$, and output $y(t)$. What function can we put into the system so that we will get the same function out, scaled by a constant? Such functions are called *eigenfunctions* and their associated constants are called *eigenvalues*. In symbols, for a system performing operation H on its input, we have $H[f(\cdot)] = \lambda f(\cdot)$, where $f(\cdot)$ is the eigenfunction and λ is its eigenvalue.[1]

Let's try $x(t) = e^{j\omega t}$. Then

$$y(t) \quad = \quad x(t) * h(t)$$

[1] *Eigen* is German for *self*, i think. Well, on good days, that is.

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau \\
&= e^{j\omega t}\int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)d\tau \\
&= e^{j\omega t}H(\omega)
\end{aligned}
$$

where $H(\omega)$ is defined as $\int_{-\infty}^{\infty} e^{-j\omega\tau)}h(\tau)d\tau$.

Note that $H(\omega)$ could be complex. That means that we can also write:

$$y(t) = |H(\omega)|e^{j(\omega t+\angle H(\omega))}$$

What if we tried a cosine? If we assume that the system performs operation H on its input:

$$
\begin{aligned}
y(t) &= H[x(t)] \\
&= H[\cos(\omega t)] \\
&= H[\frac{1}{2}(e^{j\omega t}+e^{-j\omega t})] \\
&= \frac{1}{2}H[e^{j\omega t}]+\frac{1}{2}H[e^{-j\omega t}] \\
&= \frac{1}{2}|H(\omega)|e^{j\omega t+j\angle H(\omega)}+\frac{1}{2}|H(-\omega)|e^{-j\omega t+j\angle H(-\omega)} \\
&\neq \lambda\cos(\omega t)
\end{aligned}
$$

Oh well. Because there is no guarantee that $|H(\omega)|$ is even and $\angle(H(\omega))$ is odd, we have to delete cosine from the list of candidates. Similar reasoning allows us to delete sine.

Exercise Prove that $\sin(\omega t)$ is not an eigenfunction.

Anyway, we still have an important result. If you gave me a system with impulse response $h(t)$ and input $x(t)$ and told me to find $y(t)$, i could always convolve and give you an answer. But since $e^{j\omega t}$ is an eigenfunction for the convolution operator, if i can

1. represent the input $x(t)$ as the sum of complex exponentials
2. determine $H(\omega)$ for the impulse response $h(t)$

then i can give you the output $y(t)$ as the sum of complex exponentials, scaled by H at the appropriate values of ω. That is, if

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

then

$$
\begin{aligned}
y(t) &= \sum_{n=-\infty}^{\infty} X_n H(n\omega_0)e^{jn\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} X_n |H(n\omega_0)|e^{j(n\omega_0 t+\angle H(n\omega_0))}
\end{aligned}
$$

Notice that i didn't have to do any convolution. What a feature.

5 Sinusoidal Steady-State Response of Real-World LTI Systems

If $h(t)$ is real, then we can definitely say some useful things about cosine and sine being inputs into LTI systems.

If $h(t)$ is real, then $h(t) = h^*(t)$. So

$$\begin{aligned} H^*(\omega) &= [\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt]^* \\ &= \int_{-\infty}^{\infty} h^*(t)e^{j\omega t}dt \\ &= \int_{-\infty}^{\infty} h(t)e^{-j(-\omega)t}dt \\ &= H(-\omega) \end{aligned}$$

Rewriting $H^*(\omega)$ and $H(-\omega)$ in polar form gives:

$$\begin{aligned} H^*(\omega) &= [|H(\omega)|e^{j\angle H(\omega)}]^* \\ &= |H(\omega)|e^{-j\angle H(\omega)} \\ H(-\omega) &= |H(-\omega)|e^{j\angle H(-\omega)} \end{aligned}$$

If we equate the two polar forms,

$$\begin{aligned} |H(-\omega)| &= |H(\omega)| \\ \angle H(-\omega) &= -\angle H(\omega) \end{aligned}$$

In other words, for a real $h(t)$ the magnitude of the frequency response is even and the phase of the frequency response is odd.

Let's now go back to trying a cosine as an input into an LTI system.

$$\begin{aligned} y(t) &= H[\cos(\omega t)] \\ &= \frac{1}{2}|H(\omega)|e^{j\omega t + j\angle H(\omega)} + \frac{1}{2}|H(-\omega)|e^{-j\omega t + j\angle H(-\omega)} \\ &= \frac{1}{2}|H(\omega)|e^{j\omega t + j\angle H(\omega)} + \frac{1}{2}|H(\omega)|e^{-j\omega t - j\angle H(\omega)} \\ &= |H(\omega)|\cos[\omega t + \angle H(\omega)] \end{aligned}$$

So if our impulse response is real, then a cosine as input comes back out still looking like a cosine, but its amplitude is scaled by the magnitude of the frequency response, and its phase is shifted by the phase of the frequency response. But all real-world systems are real, so this should work on any system i care to take down to the lab and throw cosines into. In fact, this gives me a good way to figure out what $H(\omega)$ is; i can generate all sorts of cosines with amplitude 1 at various frequencies and record the output cosines' amplitude and phase shift to construct $H(\omega)$. This procedure is actually used in the real world.

Exercise Prove that if $h(t)$ is real, then sine as input gives a sine as output.

6 A Look Ahead

Fourier series you have already seen in differential equations. We're going to use it as a stepping stone into the Fourier transform. After developing the FT, we'll find out that we have a shortcut to convolution. This will be exceedingly cool.