

Notes 03 largely plagiarized by %khc

1 Discrete-Time Convolution

Let's begin our discussion of convolution in discrete-time, since life is somewhat easier in that domain. We start with a signal $x[n]$ that will be the input into our LTI system H . First, we break $x[n]$ into the sum of appropriately scaled and shifted impulses, as in Figure 1(a), with the impulse function defined in discrete-time as in Figure 1(b). In Figure 1(a),

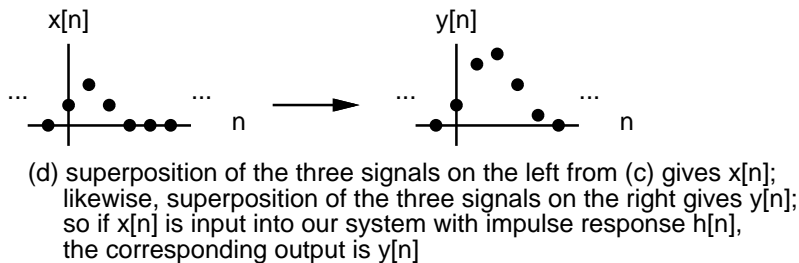
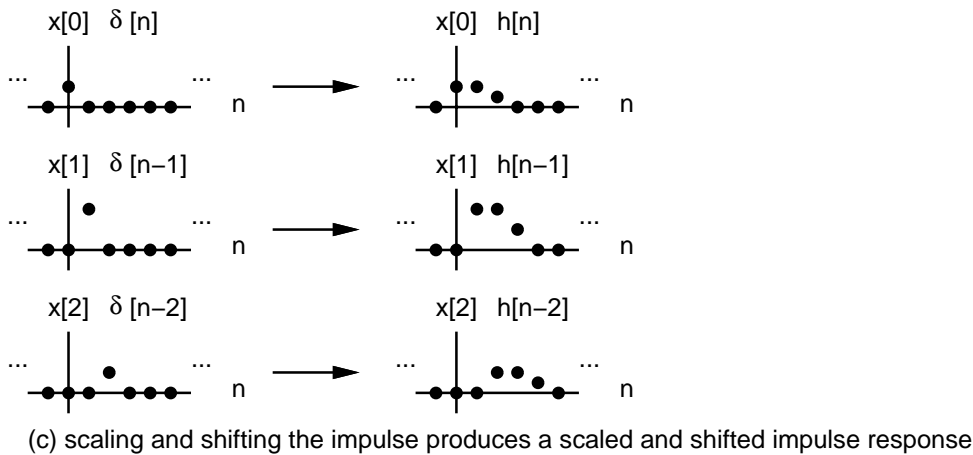
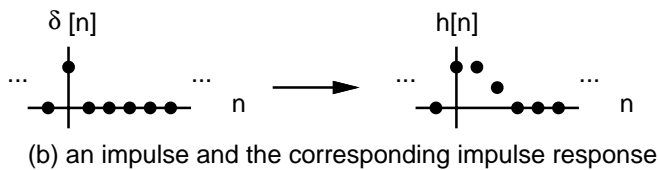
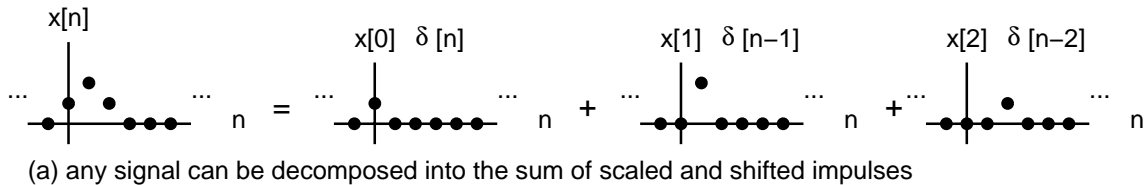


Figure 1: Discrete-time convolution.

we have decomposed $x[n]$ into the sum of $x[0]\delta[n]$, $x[1]\delta[n-1]$, and $x[2]\delta[n-2]$. In general, any $x[n]$ can be broken up into the sum of $x[k]\delta[n-k]$, where $x[k]$ is the appropriate scaling for an impulse $\delta[n-k]$ that is centered at $n=k$.

In other words, we have:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Next, we are given $h[n]$, the response of the system to an impulse centered at $n = 0$. In other words, if $\delta[n]$ is the input into this system, the output is $h[n]$. Utilizing the homogeneity property of a linear system, if $x[0]\delta[n]$ is the input, the output is $x[0]h[n]$. If $x[1]\delta[n-1]$ is the input, the output is $x[0]h[n-1]$ by both the homogeneity and time-invariant properties of H . A similar argument can be made for $x[2]\delta[n-2]$ as input producing $x[2]h[n-2]$ as output. This is illustrated in Figure 1(c).

Finally, the superposition property of a linear system permits us to add up the outputs to form the composite output to the $x[n]$ from Figure 1(a). The final output $y[n]$ is shown in Figure 1(d). This final output is just:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

2 Superposition Integral

Assume we have a linear system which performs the operation H on its input $x(t)$, giving the output $y(t)$. In symbols,

$$\begin{aligned} y(t) &= H[x(t)] \\ &= H\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} H[x(\tau)\delta(t-\tau)]d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)H[\delta(t-\tau)]d\tau \end{aligned}$$

where we have used the sifting integral to substitute for $x(t)$ in the second line, the additivity property of linearity [$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$] in the third, and the homogeneity property of linearity [$\alpha x(t) \rightarrow \alpha y(t)$] in the fourth.

What is $H[\delta(t-\tau)]$? Well, first we'd better ask what $\delta(t-\tau)$ is; it's an impulse function centered at $t = \tau$. $H[\delta(t-\tau)]$ is then the response of the system H at time t to an impulse centered at time τ . Let's denote $H[\delta(t-\tau)]$ by $g(t, \tau)$. Note that g is a function of two variables. It's a function of t because the response of the system is going to be a function of time. It's a function of τ , because the system may be time varying, so *i need to know where the impulse is centered*. If the system is time varying, the response of the system to an impulse centered at $t = t_1$ will in general be different from the response of the system to an impulse centered at $t = t_2$. Anyway, the output of the system $y(t)$ to any input $x(t)$ becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t, \tau)d\tau$$

But if the system is time-invariant, we can further massage $g(t, \tau)$ into something more palatable. Consider a time invariant system, which responds to an impulse centered at $t = a$ as in Figure 2(b); this system response is denoted $g(t, a)$. Now, because the system is time invariant, if i shift the impulse by Δ , the system response should also shift by Δ , as in Figure 2(d). This new system response is denoted $g(t-\Delta, a-\Delta)$. Why the second argument shifts is easy to see; the second argument tells us where the impulse is centered, and since the center has been shifted by Δ , the new center is $a-\Delta$. Big deal. What about the first argument? The best way to see this is to note what happens to the point (p, q) ; it gets shifted by Δ also. Now let's make $a = \Delta$. In other words, we shift the impulse to the origin by a units and the system response gets shifted, by a total of a units, as in Figure 2(f).

So the output of the system $y(t)$ to any input $x(t)$ becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau, 0)d\tau$$

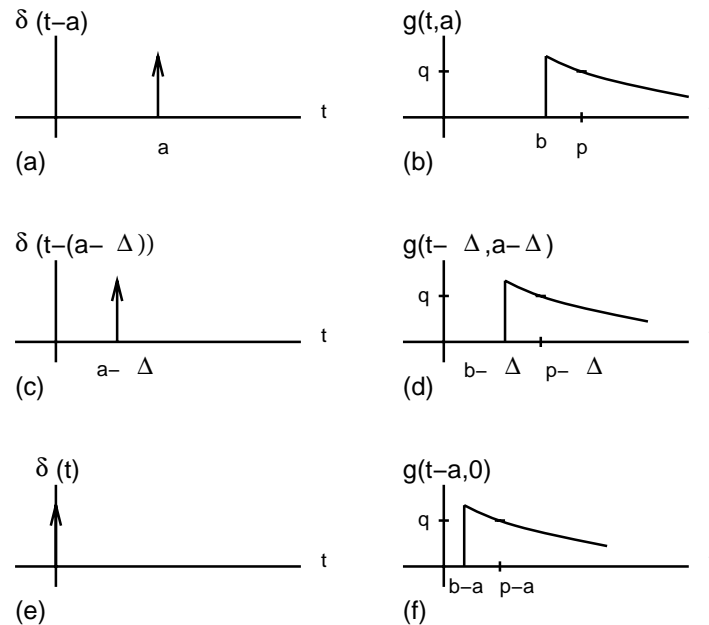


Figure 2: (a) impulse centered at $t = a$, (b) response of system to impulse centered at $t = a$, (c) impulse centered at $t = a - \Delta$, (d) response of system to impulse centered at $t = a - \Delta$, (e) impulse centered at $t = 0$, (f) response of system to impulse centered at $t = 0$. We assume that the system is time invariant, so that (c) and (d) are simply (a) and (b) shifted. For the particular choice of $a = \Delta$, we obtain (e) and (f) from (c) and (d). Note what happens to the point (p, q) in (b), (d), and (f).

We can then define a new impulse response $h(t - \tau) = g(t - \tau, 0)$ and finally we have the superposition integral as seen in lecture:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

We then note that the superposition integral conveniently corresponds to the convolution operation, as previously seen in math classes. So, to summarize:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \end{aligned}$$

But why bother? Well, if you have the impulse response $h(t)$ of any LTI system, and i give you *any* input $x(t)$, you can tell me what $y(t)$ is by doing a convolution of x and h . That means you don't have to keep on going down to the lab and putting inputs into your LTI system to figure out what your output is, since you can calculate it.

Keep in mind that the convolution integral with $h(t - \tau)$ only works for linear time invariant systems.

Of course, in real lifeTM, many systems are nonlinear. but with appropriate linearization techniques, you can approximate them by LTI systems in the regions of interest and analyze them using techniques developed in this course.

3 Useful Convolution Properties

Convolution is associative, commutative, and distributive over addition.

Exercises Prove the above statement. Use the definition of the superposition integral.

Convolution is *not* distributive over multiplication.

Exercise Consider $u(t) * u(t) = u(t) * [1u(t)] = [u(t) * 1][u(t) * u(t)]$.

Convolving $x(t)$ with:

- $\dot{\delta}(t)$ gives $\dot{x}(t)$.
- $\delta(t)$ gives $x(t)$. The identity under convolution is the unit impulse.
- $\delta(t - t_0)$ gives $x(t - t_0)$.
- $u(t)$ gives $\int_{-\infty}^t x(t) dt$.

Exercises Prove these. Of the three, the first is the most difficult, and the second the easiest.

4 Time Invariance, Causality, and BIBO Stability Revisited

Now that we have the convolution operation, we can recast the test for time invariance in a new light. If $x(t) \rightarrow y(t)$, we look to see if $x(t) * \delta(t - t_0) \rightarrow y(t) * \delta(t - t_0)$.

The impulse response $h(t)$ of a given system gives the response of that system to an impulse centered at $t = 0$. If the system is at rest and is causal, the impulse response should not begin to change from zero until it sees the impulse at $t = 0$. So an alternate way of proving causality is to determine if $h(t) = 0$ for $t < 0$.

The superposition integral also gives us another way of looking at BIBO stability. If $h(t)$ is not absolutely integrable (that is, if $\int_{-\infty}^{\infty} |h(t)| dt$ does not converge), then there is no way that $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$ is going to converge.

Exercise Is $y(t) = \int_{-\infty}^{\infty} x(\tau) x(t - \tau) d\tau$ linear? Time invariant? Causal? Memoryless?

5 An Example

Of course, theory is nice, but if you can't apply it, it is not terribly useful.

This example is worked using a cookbook approach to convolution. There are more insightful ways of doing this example. One method is hinted at in the exercise at the end of this section.

Anyway, the cookbook approach for *LTI systems only*:

1. Set up $x(\tau)$.
2. Set up $h(t - \tau)$ by flipping.
3. Determine regions of integration by dragging $h(t - \tau)$.
4. Do the integration of the *product* of $x(\tau)$ and $h(t - \tau)$.

The choice of which function is x and which is h is arbitrary. Since convolution is commutative, you'll probably want to choose the simpler one (such as the one symmetric about $t = 0$) to flip and shift. Too bad I haven't done that in my example (note also that I have used t instead of τ —it made creating the figure slightly easier).

To generate $x(\tau)$, use τ in place of t .

To generate $h(t - \tau)$ is slightly harder. First, create $h(\tau)$ from $h(t)$ by replacing t with τ . Then make $h(-\tau)$ by flipping $h(\tau)$ about $\tau = 0$. Finally, realize $h(t - \tau)$ by relabeling the points on the τ axis; $\tau = a$ becomes $\tau = t + a$. Note that by increasing t , $h(t - \tau)$ moves to the right.

At this point in time, reread the previous paragraph and try to figure out why you relabel the point $\tau = a$ as $\tau = t + a$ (hint: draw $h(0 - \tau)$ and $h(1 - \tau)$, and then generalize). Try not to confuse τ with t . The difference between t and τ is the index into h .

Determining the different regions of integration is the most difficult step. We drag $h(t - \tau)$ over $x(\tau)$ by changing the value of t (*not* τ). We then look for values of t where:

- both $h(t - \tau)$ and $x(\tau)$ are nonzero (if one of them were zero, their product would be zero, and the integral would be trivial).
- the limits on the integral will change.

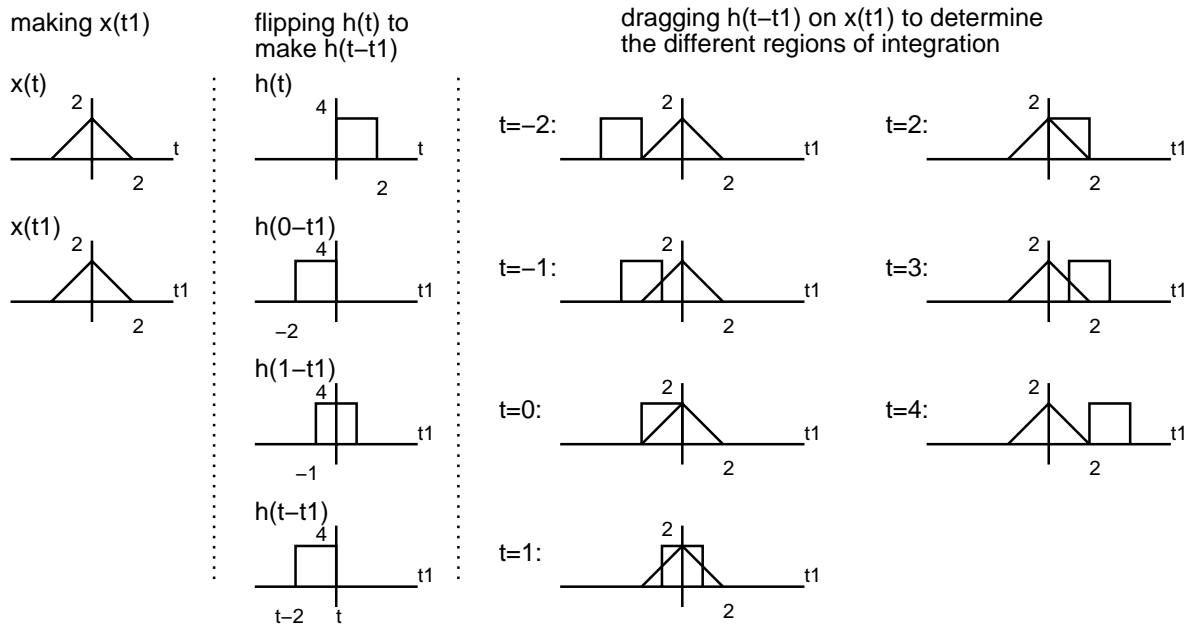


Figure 3: An example of the flip and drag convolution technique.

In my example, $h(t - \tau)$ and $x(\tau)$ are both nonzero only between $t = -2$ and $t = 4$. We have the following regions of integration:

$$y(t) = \begin{cases} 0 & \text{if } t < -2 \\ \int_{-2}^t x(\tau)h(t-\tau)d\tau & \text{if } -2 < t < 0 \\ \int_{t-2}^t x(\tau)h(t-\tau)d\tau & \text{if } 0 < t < 2 \\ \int_{t-2}^2 x(\tau)h(t-\tau)d\tau & \text{if } 2 < t < 4 \\ 0 & \text{if } t > 4 \end{cases}$$

For $-2 < t < 0$:

$$\begin{aligned} \int_{-2}^t 4(\tau+2)d\tau &= 2\tau^2 + 8\tau \Big|_{-2}^t \\ &= 2t^2 + 8t - (-8) \\ &= 2(t+2)^2 \end{aligned}$$

For $0 < t < 2$:

$$\begin{aligned} \int_{t-2}^0 4(\tau+2)d\tau + \int_0^t 4(-\tau+2)d\tau &= [2\tau^2 + 8\tau]_{t-2}^0 + [-2\tau^2 + 8\tau]_0^t \\ &= -2(t-2)^2 - 8(t-2) - 2t^2 + 8t \\ &= -4t^2 + 8t + 8 \\ &= -4(t^2 - 2t - 2) \end{aligned}$$

For $2 < t < 4$:

$$\begin{aligned} \int_{t-2}^2 4(-\tau+2)d\tau &= -2\tau^2 + 8\tau \Big|_{t-2}^2 \\ &= 2(t-2)^2 - 8(t-2) + 8 \\ &= 2(t-4)^2 \end{aligned}$$

Does this make sense? Well, the product of x and h increases from 0 at $t = -2$ to a maximum value at $t = 1$ and then falls back to zero at $t = 4$. Note that there is no discontinuity at $t = -2$, $t = 0$, $t = 2$, or $t = 4$. In general, if your x and h are piecewise continuous, y should also be continuous.

Exercise Redo this example. Then try it for the same $x(t)$, but with $h(t) = 4$ for only $-1 < t < 2$, zero elsewhere.

Exercise Redo this example. However, consider the pulse as the sum of two steps, and the triangle as the sum of three ramps. Do only one convolution of a step and a ramp, and then use linearity and time invariance to construct the whole solution.

6 A Look Ahead

More convolution fun follows. If you are uncomfortable with how to do it, try working some of the examples in the textbook or asking any of us for help.