Notes 01 largely plagiarized by %khc

1 Singularity Functions

$\delta(t)$ is the unit impulse. Of its major properties, we list the first two from which the rest can be derived:

1. $\delta(t)$ is zero everywhere except at $t = 0$.
2. $\int_{-\infty}^{\infty} \delta(t) dt = 1$.

To prove the rest of the properties, we could adopt any sort of candidate function that meets the above two criteria. For simplicity, we choose

$$\begin{aligned} \delta(t) &= \lim_{\epsilon \to 0} \delta_\epsilon(t) \\ &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \Pi(\frac{t}{2\epsilon}) \end{aligned}$$

Exercise Check that we satisfy the two properties with $\lim_{\epsilon \to 0} \delta_\epsilon(t)$.

The sampling property of $\delta(t)$ is useful: if we multiply any function by an impulse samples that function at wherever the impulse is. In symbols: $x(t)\delta(t-a) = x(a)\delta(t-a)$.

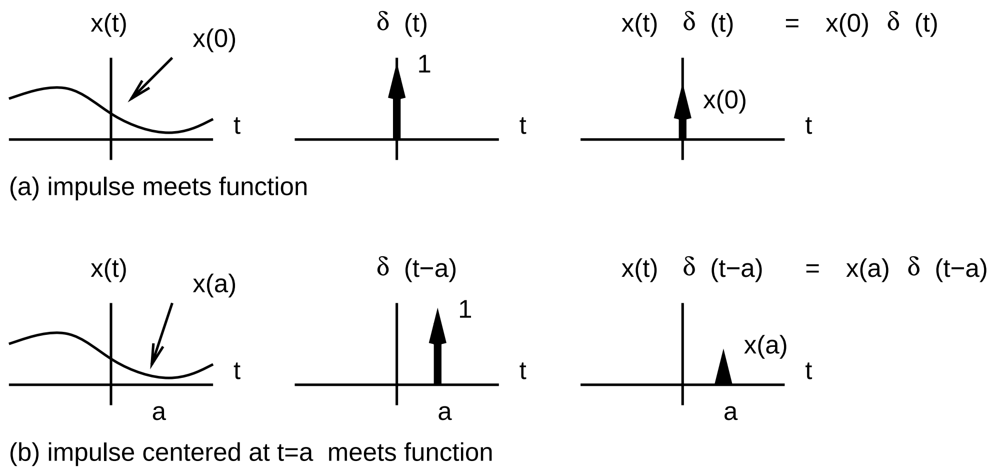

Figure 1: The sampling property of $\delta(t)$.

For the scale change property, we first draw $\delta_\epsilon(t)$ and $\delta_\epsilon(at)$. Integrating the former and taking the limit we get:

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\infty}^{\infty} \Pi(\frac{t}{2\epsilon}) dt &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} dt \\ &= 1 \end{aligned}$$

Integrating $\delta_\epsilon(at)$ and taking the limit we get:

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\infty}^{\infty} \Pi(\frac{at}{2\epsilon}) dt &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon/|a|}^{\epsilon/|a|} dt \\ &= \frac{1}{|a|} \end{aligned}$$

So we conclude $\delta(at) = \frac{1}{|a|}\delta(t)$.

For $a = -1$, we have $\delta(t) = \delta(-t)$, or the fact that $\delta(t)$ is even. Of course, we could always get this by admiring $\delta_\epsilon(t)$.

A number of our other important singularity functions are related to $\delta(t)$ by differentiation and running integration.

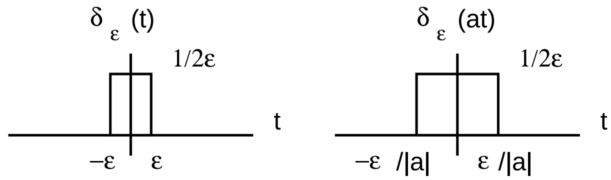

Figure 2: A scale change.

- $\frac{d}{dt}\delta(t) \stackrel{\Delta}{=} \dot{\delta}(t)$. This is also called the doublet.
- $\int_{-\infty}^{t} \delta(t)dt = u(t)$. This is the unit step, zero before $t = 0$ and unity afterwards. We are defining $u(0) = 1$, but we will be quickly handwaving around this point later on.
- $\int_{-\infty}^{t} u(t)dt = r(t)$. This is the unit ramp function, zero before $t = 0$ and t afterwards.
- $\int_{-\infty}^{t} r(t)dt = p(t)$. This is the unit parabola function, zero before $t = 0$ and $\frac{1}{2}t^2$ afterwards.

We also have $\Pi(t)$, the pulse. It is unity between $t = -\frac{1}{2}$ inclusive and $t = \frac{1}{2}$, and zero elsewhere.

In the future, when we discuss convolution, we will uncover new and exciting properties for $\dot{\delta}(t)$, $\delta(t)$, and $u(t)$.

Exercise What does $\sum_{n=-\infty}^{\infty} \delta(at - nT)$ look like? What if i multiply it by $x(t)$? What operation am i performing?

2 Sifting Integral

We have already seen the sampling property of $\delta(t)$:

$$x(t)\delta(t-a) = x(a)\delta(t-a)$$

What if i wanted to recover $x(a)$? Integration looks useful.

$$\int_{-\infty}^{\infty} x(a)\delta(t-a)dt = x(a)\int_{-\infty}^{\infty} \delta(t-a)dt = x(a)$$

Let's put everything together.

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\delta(t-a)dt &= \int_{-\infty}^{\infty} x(a)\delta(t-a)dt \\ &= x(a)\int_{-\infty}^{\infty} \delta(t-a)dt \\ &= x(a)\end{aligned}$$

So what we have said is that at any point a, if we sample $x(t)$ with an impulse centered at $t = a$ and then integrate, we will get back $x(t)$ at that point a.

What we have developed here is very similar to the sifting integral we have seen in lecture. Replacing our a with t and our t with τ, we get

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)d\tau$$

The only problem is that the stuff in the parentheses of the impulse is reversed. We can take care of that problem by noting that $\delta(t)$ is even.

Then we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

How do we interpret this equation? We have two variables here: τ and t. τ is the variable over which we integrate. If we let t be fixed to some value and do this integration, we get $x(t)$ at that given value of t. Then we can repeat this operation over and over again, changing our fixed value of t every time by some small amount. If we choose t ranging from $-\infty$ to ∞, we can then construct $x(t)$ by integrating scaled and shifted impulses.

You[1] might ask yourself the following questions: what is the point of this useless identity? If we already had $x(t)$, why bother arranging a meeting with $\delta(t)$ and an integral in order to get it back? The answer will be apparent when we begin to discuss convolution.

Exercise The stuff above might cause some brain warp. Writing it out was character-building, because i kept on confusing all the variables. Take another look at it and see if it makes sense to you.

3 Energy and Power

Energy is defined as:

$$E = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} |x(t)|^2 dt$$

Power is defined as:

$$P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt$$

Finite energy implies zero power; looking at the definition for power, we see that the integral is finite but the denominator of the fraction in the front is infinite. These are energy signals.

Power signals have infinite energy and finite power. Periodic signals are an example of this.

You are asked to find a "none of the above" signal on the problem set.

	finite energy	infinite energy
finite power	energy signal	power signal
nonexistant/infinite power	does not exist	"none of the above" signal

Figure 3: The possible types of signals depending on energy and power.

4 Energy and Power, Continued

As we will see in lecture, periodic signals have infinite energy.

$$\begin{aligned} E &= \lim_{\tau \to \infty} \int_{-\tau}^{\tau} |x(t)|^2 dt \\ &= \lim_{\frac{nT}{2} \to \infty} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} |x(t)|^2 dt \\ &= \lim_{n \to \infty} n \int_{T} |x(t)|^2 dt \\ &= \infty \end{aligned}$$

However, they have finite power:

$$\begin{aligned} P &= \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \\ &= \lim_{\frac{nT}{2} \to \infty} \frac{1}{nT} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} |x(t)|^2 dt \\ &= \lim_{n \to \infty} \frac{n}{nT} \int_{T} |x(t)|^2 dt \end{aligned}$$

[1] Yes, you, the astute student. On the other hand, i'm still on vacation, and my brain is floating around the bay right now.

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{T}\int_T |x(t)|^2 dt \\
&= \frac{1}{T}\int_T |x(t)|^2 dt
\end{aligned}
$$

where the notation $\int_T$ means to take the integral over the period (who cares what the limits are, as long as you integrate over a period; this lets you choose sometimes simpler integrals to perform).

So periodic signals are power signals, since they have infinite energy and finite power.

Exercises Categorize the following signals: $t^2[u(t+2) - u(t-2)]$, $e^{-t}u(t)$, $\sum_{n=-\infty}^{\infty}[u(t-n) - u(t-n-0.1)]$. What is the energy and power contained in each signal?

5 A Look Ahead

Convolution, a nice and confusing topic that you may already have seen in differential equations, is up within the next two weeks. Please review the sifting integral and the various system definitions we will learn on next Wednesday in preparation. These three topics will be needed to develop convolution.