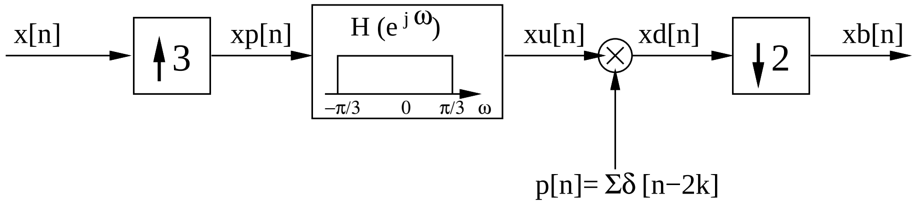

Fig. 1. Block diagram of up-sample by 3, interpolate, re-sample at twice the sample period, and then downsample by a factor of 2.

In this example, an input discrete time signal $x[n]$ (representing samples of a continuous time signal $x(t)$ at $T = 1/4$sec) is converted to $x_b[n]$, representing samples of a continuous time signal $x(t)$ with sampling period $T = 1/6$ sec. The block diagram in Figure 1, shows the processing steps involved.

Let $x(t) = \cos(2\pi t)$ with sampling period $T_s = 0.25$ sec, and $x[n] = \cos(\pi n/2)$.

The CTFT for the sampled signal is calculated in the usual way:

$$x_\delta(t) \;=\; \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

The CT spectrum for the sampled signal is:

$$X_\delta(j\omega') = \; X(j\omega') * \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega' - \frac{k2\pi}{T_s})$$

or

$$X_\delta(j\omega') \;=\; \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j\omega' nT_s}$$

To calculate the normalized DTFT, let $\omega = \omega' T_s$, then

$$X(e^{j\omega}) \;=\; \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X_\delta(\frac{j\omega}{T_s})$$

.

1 Upsample

To upsample $x[n]$ by a factor of 3, two extra samples are added in between every original sample, thus $x_p[0] = x[0], x_p[3] = x[1], x_p[6] = x[2], ...$, then:

$$\begin{aligned} X_p(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_p[n] e^{-j\omega n} && (1)\\ &= ... + x_p[0]e^{-j0} + x_p[3]e^{-j3\omega} + x_p[6]e^{-j6\omega} + ... && (2)\\ &= ... + x[0]e^{-j0} + x[1]e^{-j3\omega} + x[2]e^{-j6\omega} + ... && (3)\\ & && (4) \end{aligned}$$

Thus $X_p(e^{j\omega}) = X(e^{j3\omega})$. (See Fig. 2.)

An interpolation filter $h[n]$ is used to fill in intermediate values, with

$$h[n] = 3\frac{\sin(n\pi/3)}{\pi n}.$$

Note that $h[0] = 1$ so that the original sample points will have the original heights, i.e. $x_u[0] = x[0] = x_p[0]$. The spectrum of the interpolated signal is $X_u(e^{j\omega}) = X_p(e^{j\omega})H(e^{j\omega})$.

The resulting spectrum $X_u(e^{j\omega})$ is equivalent to the DTFT of the original cosine sampled at $T_u = 1/12$ sec.

2 Down sample

To down sample by a factor of 2, first every other sample is set to zero (giving $x_d[n]$), and finally half the samples are discarded, giving $x_b[n]$. Here, the final output $x_b[n]$ is equivalent to sampling the original cosine at $T_d = 1/6$ sec.

Subsampling $x_u[n]$ by 2 is equivalent to multiplying by a pulse train $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2k]$. The DTFT spectrum is calculated from the circular convolution:

$$p[n] \cdot x_u[n] \rightarrow^{DTFT} \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X_u(e^{j(\omega-\theta)}) d\theta = X_d(e^{j\omega})$$

where $P(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - k\pi)$.

Finally, by discarding odd samples, $x_b[n] = x_d[2n]$ and hence $X_b(e^{j2\omega}) = X_d(e^{j\omega})$. The figure shows how each frequency component is scaled in frequency.

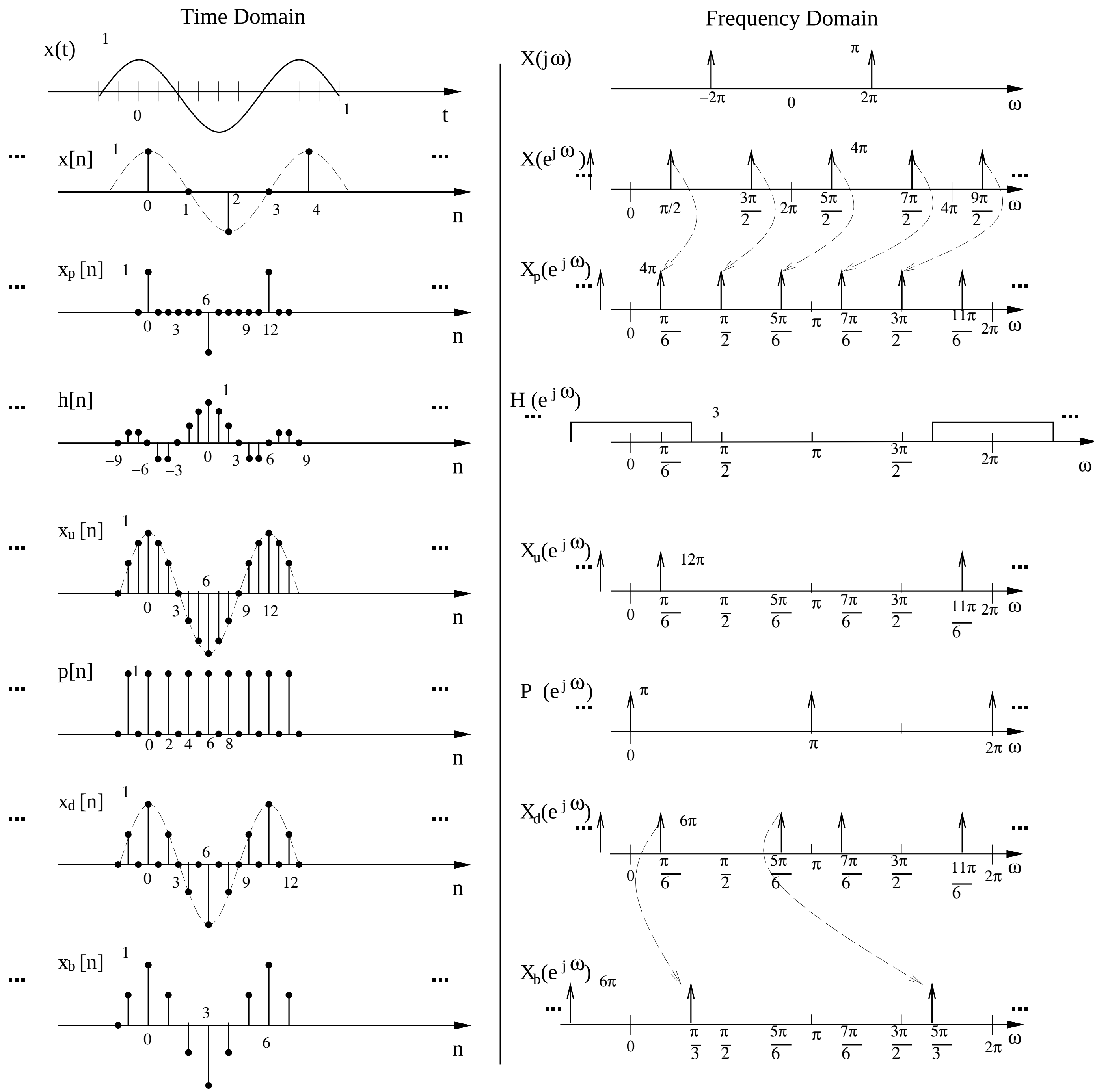

Figure 2. Time and frequency plots for combined up/down sample example.