Section 2

Monday, July 1

CS 70: Discrete Mathematics and Probability Theory, Summer 2013

Your TA will give a crash course in big O notation.

1. Determine whether each of the following is true or false. (You do not need to give formal proofs.)

   1a. $n^3 + 10n^2 \in O(n^4)$

   1b. $n \cdot \log_2 n \in O(50n)$

   1c. $\log_2^2 n \in O(\log_{10}^2 n)$

   1d. $n^5 \in O(1.1^n)$

2. A guest at a party is a *celebrity* if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party (since if there were two, they would know each other). A party may or may not have a celebrity. Your assignment is to find the celebrity at a party, if one exists, by asking only one type of question — asking a guest whether they know a second guest. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer truthfully.

   2a. Suppose you know the party has a celebrity. Use induction to show that if there are n people at the party (not including yourself, the question asker), then you can identify the celebrity by asking $n - 1$ questions. [*Hint*: First ask a question to eliminate one person as a possible celebrity. Then use the inductive hypothesis to identify the celebrity from the remaining party members.]

   2b. Now consider the general case, where the party might not have a celebrity. Modify your induction to show that you can find the celebrity, or verify that none exists, with $\leq 3(n - 1)$ questions. [*Hint*: You may need to ask an additional two questions in each inductive step in order to verify that your possible celebrity is actually a celebrity.]

3. Recall that the Fibonacci numbers $f_0, f_1, f_2, \ldots$ are defined by the equations $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2}$$

   for $n = 2, 3, 4, \ldots$.

   3a. Use induction to show that $3 \mid f_{4n}$, for all n.

   3b. Use strong induction to show that every natural number can be written as a sum of distinct Fibonacci numbers (for example, $381 = 1 + 5 + 13 + 89 + 233$).[1]

[1] See `http://en.wikipedia.org/wiki/Fibonacci_coding` for an application of this result.

4. (Optional challenge) We know that every positive rational number can be written as a quotient of products of primes. Now prove that every positive rational number can be written as a quotient of products of *factorials* of primes.[2] For example, $\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$.

   [*Hint*: Recall that every integer has a unique prime factorization. Let p be the largest prime appearing in the factorizations of the numerator and denominator, e.g. $\frac{10}{9} = \frac{2 \cdot 5}{3 \cdot 3}$ so $p = 5$ in this case. Use strong induction on p.]

[2] This was problem B1 on the 2009 William Lowell Putnam Mathematics Competition.