Homework 7

Due: Tuesday, August 13, 4:00pm

CS 70: Discrete Mathematics and Probability Theory, Summer 2013

1. [6 points] You have a random variable X whose expectation you'd like to estimate. You have the ability to draw samples from X's distribution, and you know nothing about $E(X)$, and you only know that $\text{Var}(X) \leq 10$. (You may assume $\text{Var}(X) = 10$, since this is the worst case.) To estimate $E(X)$, you take n i.i.d. samples of X and average them. Using the bound proved in lecture based on Chebyshev's Inequality (equation (3) in lecture note 17):

   1a. [2 points] Suppose you take 1000 samples. How confident are you that your estimate is within an absolute error of 0.5?

   1b. [2 points] Suppose instead that you want an absolute error of at most 2 and a confidence parameter of 0.02 (you want to be "98% confident"). How many samples do you need?

   1c. [2 points] Suppose instead that you take 2500 samples and you want a confidence parameter of 0.1 ("90% confident"). What absolute error bound will you get with this confidence?

2. [8 points] Suppose you have a random variable X with distribution $\text{Geom}(p)$, and you'd like to estimate p.

   2a. [4 points] Since $E(X) = \frac{1}{p}$, a natural approach to estimate p is to get a sample from X and then take the reciprocal $\frac{1}{X}$. Show that this is *not* an unbiased estimator for p.

   2b. [4 points] Suppose you take n i.i.d. samples $X_1, \ldots, X_n$, and you let Y be the fraction of these samples that equal 1. Show that Y *is* an unbiased estimator for p.

3. [8 points] I am playing in a tennis tournament, and I am up against a player I have watched but never played before. Based on what I have seen, I consider three possible models for our relative strengths:

   - Model A: We are evenly matched, so that each of us is equally likely to win each game.
   - Model B: I am slightly better, so that I win each game independently with probability 0.6.
   - Model C: My opponent is slightly better and wins each game independently with probability 0.6.

   Before we play, I consider each of these possibilities to be equally likely. In our match, we play until one player wins three games. I win the second game, but my opponent wins the first, third, and fourth games. After the match, what is the posterior probability of model C (i.e., that my opponent is slightly better than me)?

4. [8 points] Recall the communication problem we considered in the lecture (see lecture note 18). We considered the case when X is equally likely to be 1 and 0. Now suppose $\Pr[X = 1] = q$. Re-derive the maximum a posteriori (MAP) decision rule in this more general case. Make your rule as explicit as possible. Is it still the simple majority rule we derived in class? For $q = 0.1$, $p = 0.25$, and $n = 5, 10, 15$, work out explicitly what the decision rule is. How does it compare to the simple majority rule? Does it make intuitive sense?

5. [8 points] Alice and Bob agree to try to meet for lunch between 12pm and 1pm at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch? Phrase your solution using the language of continuous random variables introduced in lecture note 19.

6. [12 points] Let X be a continuous random variable whose pdf is cx^3 (for some constant c) in the range $0 \le x \le 1$, and is 0 outside this range.

   6a. [3 points] Find c.

   6b. [3 points] Find $\Pr\left[\frac{1}{3} \le X \le \frac{2}{3} \;\middle|\; X \le \frac{1}{2}\right]$.

   6c. [3 points] Find $\mathrm{E}(X)$.

   6d. [3 points] Find $\mathrm{Var}(X)$.

7. [11 points] In class, a random variable X is specified by its *distribution* in the discrete case and by its *probability density function (pdf)* in the continuous case. To unify the two cases, we can define the *cumulative distribution function (cdf)* F, which is valid for both discrete and continuous random variables X, as follows:

$$F(a) = \Pr[X \le a], \qquad a \in \mathbb{R}.$$

   7a. [4 points] In the discrete case, show that the cdf of a random variable contains exactly the same information as its distribution, by expressing F in terms of the distribution and expressing the distribution in terms of F. For simplicity, you may assume that the discrete random variable only takes on integer values.

   7b. [4 points] In the continuous case, show that the cdf of a random variable contains exactly the same information as its pdf, by expressing F in terms of the pdf and expressing the pdf in terms of F.

   7c. [3 points] Identify two key properties that a cdf of any random variable has to satisfy.

8. [11 points] Discrete and continuous random variables have a lot of similarities but some differences too.

8a. [4 points] Suppose X is a discrete random variable. Let $Y = cX$ for some constant c. Express the distribution of Y in terms of the distribution of X.

8b. [4 points] Suppose X is a continuous random variable. Let $Y = cX$ for some constant c. Express the pdf of Y in terms of the pdf of X. Is there any difference with the discrete case? (Hint: work with cdf's introduced in problem 7.)

8c. [3 points] If $X = N(\mu, \sigma^2)$, what is the density of $Y = cX$?

9. [15 points] We begin by proving two very useful properties of the exponential distribution. We then use them to solve a problem in digital photography.

9a. [3 points] Let random variable X have exponential distribution with parameter λ. Show that, for any positive s, t, we have

$$\Pr\left[X > s + t \;\middle|\; X > t\right] = \Pr[X > s].$$

This is the "memoryless" property of the exponential distribution. We already saw the analogous memoryless property of the geometric distribution in the section 11 worksheet.

9b. [3 points] Let random variables X_1, X_2 be independent and exponentially distributed with parameters λ_1, λ_2. Show that the random variable $Y = \min\{X_1, X_2\}$ is exponentially distributed with parameter $\lambda_1 + \lambda_2$. (Hint: work with cdf's.)

9c. [5 points] You have a digital camera that requires two batteries to operate. You purchase n batteries, labeled $1, 2, \ldots, n$, each of which has a lifetime that is exponentially distributed with parameter λ and is independent of all the other batteries. Initially you install batteries 1 and 2. Each time a battery fails, you replace it with the lowest-numbered unused battery. At the end of this process you will be left with just one working battery. What is the expected total time until the end of the process? Justify your answer.

9d. [4 points] In the scenario of part 9c, what is the probability that battery i is the last remaining working battery, as a function of i?

10. [8 points] Suppose a set of final grades for a course are approximately normally distributed with a mean of 64 and a standard deviation of 7.1. (You are free to use the facts that $\Pr[N(0,1) \leq 1.3] \approx 0.9$ and $\Pr[N(0,1) \leq 1.65] \approx 0.95$.)

10a. [4 points] Find the lowest passing grade if the bottom 5% of the students fail the class.

10b. [4 points] Find the highest B+ if the top 10% of the students are given A's or A-'s.

11. [5 points] Suppose you roll a standard die 2000 times and let X be the sum of the values you get. Using the Central Limit Theorem, for what value of a is $\Pr[X \geq a] \approx \Pr[N(0,1) \geq 2]$? Justify your answer.