Homework 5

Due: Tuesday, July 30, 4:00pm

CS 70: Discrete Mathematics and Probability Theory, Summer 2013

1. [4 points] Suppose you arrange 12 different books on a bookshelf, uniformly at random. Three of the books are about discrete math, four of the books are about data structures, and the other five are about underwater basket weaving. What is the probability that the three discrete math books are all together?

2. [9 points] We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

   2a. [3 points] Find the probability that doubles were rolled.

   2b. [3 points] Given that the roll resulted in a sum of 4 or less, find the conditional probability that doubles were rolled.

   2c. [3 points] Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

3. [8 points] This problem will give you practice with conditional probability.

   3a. [4 points] I have a bag containing either a $5 bill (with probability 1/3) or a $10 bill (with probability 2/3). I then add a $5 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag (without looking in the bag). Suppose it turns out to be a $5 bill. If a second student draws the remaining bill from the bag, what is the probability that it too is a $5 bill? Show your calculations.

   3b. [4 points] Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your calculations.

4. [8 points] A murder has been committed in a city. The police are confident that the murderer must be one of the one million adult residents of the city and its surrounding area, but they initially have no reason to suspect anyone ahead of anyone else. The only piece of evidence is a DNA sample obtained from the scene. During a routine check, this sample is found to match the DNA of a man in the city (which had been collected for unrelated reasons), and

the man is put on trial for murder. At the trial, an expert witness testifies that if the man is innocent, the probability of a DNA match is $1/10,000$, and as a result the man is convicted. You may assume that, if the man is guilty, then his DNA matches with probability 1. In this problem we will see whether this verdict was justified.

4a. [4 points] Let M denote the event that the DNA matches, and I the event that the man is innocent. From the above data, write down $\Pr[I]$, $\Pr[\overline{I}]$, $\Pr[M \mid I]$, and $\Pr[M \mid \overline{I}]$.

4b. [4 points] Use the Inference Rule to compute the probability that the man is innocent, given that the DNA matched. Do you think the man should have been convicted?

Note: This is an example of the so-called "Prosecutor's Paradox", whereby a judge or jury confuses $\Pr[M \mid I]$ (the probability that the evidence points to an innocent person) with $\Pr[I \mid M]$ (the probability that the accused is innocent given the evidence), and assumes that if the former is small then the latter is also small.

5. [9 points] You are given a bag containing four marbles and are told that the number of blue marbles in the bag is equally likely to be any number between 0 and 4 inclusive; the other marbles are yellow (Cal colors!). So, in particular, the (prior) probability of having any particular number of blue marbles is $1/5$. Suppose you pick a marble at random from the bag and it is blue. Use the general form of the Inference Rule to calculate the posterior probability for each possible number of blue marbles originally in the bag. Do the answers look intuitively reasonable?

Guidance: For each $i \in \{0, 1, 2, 3, 4\}$, let A_i be the event that there were exactly i blue marbles to begin with (and note that the A_i's partition Ω), and let B be the event that the chosen marble is blue. This problem is asking you to compute the posterior probability $\Pr[A_i \mid B]$ for each i.

6. [7 points] In this problem we will see that pairwise independence of events does not imply mutual independence. Two fair dice are thrown. Let A be the event that the number on the first die is odd, B the event that the number on the second die is odd, and C the event that the sum of the two numbers is odd.

6a. [4 points] Show that the three events A, B, C are pairwise independent (i.e., each pair (A, B), (B, C), and (A, C) is independent).

6b. [3 points] Show that the events A, B, C are not mutually independent (also known as *fully independent*).

7. [8 points] This problem will give you practice computing probabilities of unions of events.

7a. [4 points] You have a bin that initially has 10 red balls in it. You remove a ball at random and replace it with a new, purple ball. Again, you remove a ball at random from the bin and replace it with a new, purple ball, and you repeat this process until you have picked a total of 5 balls out of the bin. (Note that the bin always has 10 balls in it.) Give an exact expression for the probability that at least one of the balls you pick out of the bin is purple. (Hint: Try complementing the event and then using the product rule.)

7b. [4 points] In a group of ten people, seven can play the piano, five can play the saxophone, four can play the violin, four can play the piano and the saxophone, three can play the piano and the violin, two can play the saxophone and the violin, and one person can play all three instruments. Suppose a person is picked uniformly at random from the group. Use the inclusion-exclusion principle to calculate the probability that this person can play at least one instrument. Explain your calculation clearly.

8. [9 points] Mr. and Mrs. Smith decide to continue having children until they either have their first girl or until they have five children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let B and G denote the numbers of boys and girls (respectively) that the Smiths have.

8a. [3 points] Write down the sample space together with the probability of each outcome.

8b. [3 points] Write down the distributions of the random variables B and G.

8c. [3 points] Compute the expectations of B and of G using a direct calculation.

9. [16 points] Solve each of the following problems using linearity of expectation. Clearly explain your methods.

9a. [4 points] A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

9b. [4 points] A building has n floors numbered $1, 2, \ldots, n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the expected number of floors the elevator stops at (not counting the ground floor)?

9c. [4 points] A coin with Heads probability p is flipped n times. A "run" is a maximal sequence of consecutive flips that are all the same. (Thus, for example, the sequence HTHHHTTH with $n = 8$ has five runs.) Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Justify your calculation carefully.

9d. [4 points] In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability 1/3 (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability 1/5, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

10. [8 points] The well-known Bubblesort algorithm sorts a list $a_1, a_2, \ldots, a_n$ of numbers by repeatedly swapping adjacent numbers that are inverted (i.e., in the wrong relative order) until there are no remaining inversions. (Note that the number of swaps required does not depend on the order in which the swaps are made.) Suppose that the input to Bubblesort is a random permutation of the numbers $a_1, a_2, \ldots, a_n$, so that all $n!$ orderings are equally likely, and that all the numbers are distinct. What is the expected number of swaps performed by Bubblesort?

11. [6 points] In lecture we mentioned that if X, Y are independent random variables, then $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$. In this problem we will see that the converse is not true in general. Consider the following random variables X and Y, both defined on a probability space with 3 equally likely outcomes:

    On the 1$^{\text{st}}$ outcome, let $X = 1$ and $Y = 0$.
    On the 2$^{\text{nd}}$ outcome, let $X = 0$ and $Y = 1$.
    On the 3$^{\text{rd}}$ outcome, let $X = 1$ and $Y = 2$.

    11a. [3 points] Show that $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ by direct calculations.

    11b. [3 points] Show that X and Y are not independent.

12. [8 points] James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe, and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces temporary amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with equal probabilities. On average, how long does it take before he realizes that the door is unlocked and escapes?

    Hint: Let the random variable T denote the time to escape. By the Total Expectation Rule, you can write the $\mathrm{E}(T) = \mathrm{E}(T \mid A)\Pr[A] + \mathrm{E}(T \mid S)\Pr[S] + \mathrm{E}(T \mid D)\Pr[D]$, where A, S, D are the events that Bond chose to go through the AC-duct, the sewer, and the door respectively on his first attempt, and $\mathrm{E}(T \mid A)$ means the expected time to escape given that Bond went through the AC-duct, etc. To solve the problem, think how to express each of $\mathrm{E}(T \mid A)$, $\mathrm{E}(T \mid S)$, and $\mathrm{E}(T \mid D)$ in terms of $\mathrm{E}(T)$.