Homework 4

Due: Tuesday, July 23, 4:00pm

CS 70: Discrete Mathematics and Probability Theory, Summer 2013

1. [4 points] In the De Bruijn graph for $k = 10$, what is the length of a shortest cycle (i.e., with the fewest edges) containing the node 111000011? Justify your answer.

2. [6 points] One of the CS70 TAs lives in a building with an electronic garage door that is opened by a remote control. One day, you overhear the TA's roommate talking about how, after losing the remote control and having the landlord demand \$60 to replace it, they discovered that the remote controls can be bought online for \$12, and all you need to do after buying one is to open it up and set ten tiny on-off switches to match the "code" of the door. These settings, of course, can be obtained by looking at the switches in another remote for the same door. After a particularly nasty homework, you decide to take revenge by getting one of these remotes and breaking into the garage. Of course, unlike the roommate, you dont have access to another remote from which to copy the code. It takes 5 seconds to flip a single switch in either direction (they're tiny, and EECS students arent known for fine motor control), and 1 second to test whether the current switch settings work by just pressing the button. What is the shortest amount of time that you need to definitely open the garage door? Explain precisely how you would go about achieving this minimum time. Keep in mind that, per Murphy's law, even if you try all but one of the combinations, it is possible that the right combination is the one you haven't tried.

3. [10 points] An undirected graph is called *bipartite* if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L with a vertex in R (i.e., there is no edge connecting two vertices in L or two vertices in R).

   3a. [3 points] Prove that a bipartite graph has no cycles of odd length.

   3b. [3 points] Prove that $\sum_{v \in L} \text{degree}(v) = \sum_{v \in R} \text{degree}(v)$.

   3c. [2 points] Let s denote the average degree of vertices in L and t the average degree of vertices in R, i.e., $s = \frac{1}{|L|} \sum_{v \in L} \text{degree}(v)$ and $t = \frac{1}{|R|} \sum_{v \in R} \text{degree}(v)$. Prove that $s/t = |R|/|L|$.

   3d. [2 points] In 1992, the University of Chicago interviewed a random sample of 2500 people in the U.S. about the number of opposite-gender sex partners they had had. They reported that on average men have 74% more opposite-gender partners than women. At around the same time, the U.S. Census Bureau reported that the female population of the U.S. was about 140 million and the male population was about 134 million. With reference to part c, explain why the University of Chicago and the U.S. Census Bureau can't both be right.

4. [30 points] The only way to learn counting is to practice, practice, practice. You should leave your answers as tidy expressions involving factorials, binomial coefficients etc., rather than evaluating them as decimal numbers. Also, you should explain clearly how you arrived at your answers; bald solutions with no explanation will receive no credit.

    4a. [2 points] How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

    4b. [2 points] How many different 13-card bridge hands are there that contain no aces?

    4c. [2 points] How many different 13-card bridge hands are there that contain all four aces?

    4d. [2 points] How many different 13-card bridge hands are there that contain exactly 5 spades?

    4e. [2 points] Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

    4f. [2 points] How many 17-bit strings are there that contain exactly 6 ones?

    4g. [2 points] How many 66-bit strings are there that contain more ones than zeros?

    4h. [2 points] How many different anagrams of KENTUCKY are there? (An anagram of KENTUCKY is any reordering of the letters of KENTUCKY, i.e., any string made up of the letters K, E, N, T, U, C, K and Y, in any order. The anagram does not have to be an English word.)

    4i. [2 points] How many different anagrams of ALASKA are there?

    4j. [2 points] How many different anagrams of CALIFORNIA are there?

    4k. [2 points] How many different anagrams of MISSISSIPPI are there?

    4l. [2 points] We have 8 balls, numbered 1 through 8, and 24 bins. How many different ways are there to distribute these 8 balls among the 24 bins?

    4m. [2 points] How many different ways are there to throw 8 identical balls into 24 bins?

    4n. [2 points] We throw 8 identical balls into 5 bins. How many different ways are there to distribute these 8 balls among the 5 bins such that no bin is empty?

    4o. [2 points] There are 30 students currently enrolled in a class. How many different ways are there to pair up the 30 students, so that each student is paired with one other student?

5. [8 points] Suppose you zealously learn 35 new programming languages, and you wish to demonstrate your prowess. Your boss assigns you 10 tasks, each of which requires writing a program. This is your time to shine.

    5a. [2 points] In how many ways can you choose a language for each program, if you insist that no two programs be written in the same language?

    5b. [2 points] In the above scenario, how many possibilities are there for which set of languages you get to use?

5c. [2 points] Now suppose you do *not* insist that no two programs be written in the same language. In how many ways can you choose a language for each program?

5d. [2 points] Continuing from part c, you want to brag to your friends by telling them how many programs you wrote in each of the 35 languages. How many possibilities are there for this information?

6. [12 points] Let p be a prime number and let k be a positive integer.

6a. [4 points] We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable. We have a piece of string. We want to make a pretty decoration by threading p many beads onto the string (from left to right). We can choose any sequence of colors, subject only to one rule: the p beads must not all have the same color. How many different ways are there to construct such a sequence of beads?

6b. [4 points] Now we tie the two ends of the string together, forming a circular necklace. This lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors — red (R), green (G), and blue (B) — then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.) Count how many non-equivalent necklaces there are, if the p beads must not all have the same color. (Hint: What can you conclude if rotating all the beads on a necklace to another position produces an identical looking necklace? Recall that p is prime.)

6c. [4 points] How can you use the above reasoning to give a new proof of Fermat's Little Theorem? (Recall that the theorem says that if $a \not\equiv 0 \pmod{p}$ then $a^{p-1} \equiv 1 \pmod{p}$.)

7. [10 points] Consider the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

7a. [5 points] Prove the identity by algebraic manipulation (using the formula for binomial coefficients).

7b. [5 points] Prove the identity using a combinatorial argument.

8. [10 points] Your friend proposes the following game. She will roll six fair dice. If the number of different numbers that show up is exactly four, then you win $1 from her. Otherwise, she wins $1 from you. Would you play this game? Justify your answer with a calculation. (Be very careful in counting the number of ways that exactly four distinct numbers can show up! You should find that the game is rather finely balanced.)

9. [10 points] You have two boxes. The first box contains two identical red balls and one blue ball, and the second box contains one red ball and two identical blue balls. You pick the first box with probability 1/4 and the second box with probability 3/4, and then you draw two balls uniformly at random without replacement from your chosen box. That is, you pull out one ball at random, and then pull out another ball at random from the same box, without putting the first ball back.

   9a. [5 points] Compute the probability of each outcome for this experiment. An outcome should specify which box was chosen, the color of the first ball, and the color of the second ball, so you should find that there are 8 different outcomes.

   9b. [5 points] Compute Pr[the two chosen balls have different colors].