Homework 1

Due: Tuesday, July 2, 4:00pm

CS 70: Discrete Mathematics and Probability Theory, Summer 2013

1. [12 points] Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

   1a. [3 points] $P \wedge (Q \vee P) \;\equiv\; P \wedge Q$

   1b. [3 points] $(P \Rightarrow Q) \Rightarrow R \;\equiv\; P \Rightarrow (Q \Rightarrow R)$

   1c. [3 points] $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) \;\equiv\; P \Rightarrow (Q \Rightarrow R)$

   1d. [3 points] $(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \;\equiv\; (Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$

2. [9 points] Decide whether each of the following propositions is true, when the domain for x and y is the real numbers $\mathbb{R}$. Prove your answers.

   2a. [3 points] $\forall x \exists y\ (xy \geq x^2)$

   2b. [3 points] $\exists y \forall x\ (xy \geq x^2)$

   2c. [3 points] $\neg \forall x \exists y\ (xy > 0 \Rightarrow y > 0)$

3. [9 points] Determine whether the following equivalences hold, and give brief justifications for your answers. Clearly state whether or not each pair is equivalent.

   3a. [3 points] $\neg \forall x\ \exists y\ \big(P(x) \Rightarrow \neg Q(x,y)\big) \;\equiv\; \exists x\ \forall y\ \big(P(x) \wedge Q(x,y)\big)$

   3b. [3 points] $\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big) \;\equiv\; \forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$

   3c. [3 points] $\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big) \;\equiv\; \forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$

4. [7 points] Here is an extract from Lewis Carroll's treatise *Symbolic Logic* of 1896:

   (I) No one, who is going to a party, ever fails to brush his or her hair.

   (II) No one looks fascinating, if he or she is untidy.

   (III) Opium-eaters have no self-command.

   (IV) Everyone who has brushed his or her hair looks fascinating.

   (V) No one wears kid gloves, unless he or she is going to a party.

   (VI) A person is always untidy if he or she has no self-command.

4a. [3 points] Write each of the above six sentences as a quantified proposition over the universe of all people. You should use the following symbols for the various elementary propositions: $P(x)$ for "x goes to a party", $B(x)$ for "x has brushed his or her hair", $F(x)$ for "x looks fascinating", $U(x)$ for "x is untidy", $O(x)$ for "x is an opium-eater", $N(x)$ for "x has no self-command", and $K(x)$ for "x wears kid gloves".

4b. [2 points] Now rewrite each proposition equivalently using the contrapositive.

4c. [2 points] You now have twelve propositions in total. What can you conclude from them about a person who wears kid gloves? Explain clearly the implications you used to arrive at your conclusion.

5. [15 points] Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

5a. [3 points] For all natural numbers n, if n is odd then $n^2 + 2n$ is odd.

5b. [3 points] For all natural numbers n, $n^2 + 7n + 1$ is odd.

5c. [3 points] For all real numbers a, b, if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.

5d. [3 points] For all real numbers r, if r is irrational then $r + 1$ is irrational.

5e. [3 points] For all natural numbers n, $10n^2 > n!$.

6. [10 points] Suppose you are given a list of positive real numbers $x_1, x_2, \ldots, x_n$ with $\sum_{i=1}^{n} x_i = 1$. (Note: You can think of this as a list of probabilities! We will talk about probability theory in great detail later in the course.) Prove the following proposition:

$$\left(\forall i \in \{1, \ldots, n\} : \; x_i \leq \tfrac{2}{3}\right) \;\Leftrightarrow\; \left(\exists S \subseteq \{1, \ldots, n\} : \; \tfrac{1}{3} \;\leq\; \left(\textstyle\sum_{i \in S} x_i\right) \;\leq\; \tfrac{2}{3}\right)$$

In other words, (each number is $\leq \frac{2}{3}$) iff (you can add up some of the numbers to get a value in the range $\frac{1}{3}$ to $\frac{2}{3}$). Since this is an "if and only if", you will need to prove two implications! Clearly state which proof techniques you use.

7. [12 points] This problem is on induction.

7a. [4 points] Use simple induction to prove that for every integer $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} \;=\; \frac{n}{n+1}$$

7b. [4 points] Use simple induction to prove that for every $n \in \mathbb{N}$, $\;5 \mid (8^n - 3^n)$.

7c. [4 points] You are in a foreign country that has only 3-cent and 7-cent coins, and you have an unlimited supply of both types. Use strong induction to prove that for every integer $c \geq 12$, it is possible to form c cents exactly using these coins. How many base cases do you need?

8. [12 points] Let $P(k)$ be a predicate involving a natural number k. Suppose you know only that $(\forall k \in \mathbb{N})\ (P(k) \Rightarrow P(k+2))$ is true. For each of the following propositions, say whether the proposition is (i) definitely true, (ii) definitely false, or (iii) possibly (but not necessarily) true. Give a *brief* (one or two sentences) explanation for each of your answers.

   8a. [2 points] $(\forall n \in \mathbb{N})(P(n))$.

   8b. [2 points] $(\forall n \in \mathbb{N})(\neg P(n))$.

   8c. [2 points] $P(0) \Rightarrow (\forall n \in \mathbb{N})(P(n+2))$.

   8d. [2 points] $(P(0) \wedge P(1)) \Rightarrow (\forall n \in \mathbb{N})(P(n))$.

   8e. [2 points] $(\forall n \in \mathbb{N})(P(n) \Rightarrow ((\exists m \in \mathbb{N})(m > n + 2013\ \wedge\ P(m))))$.

   8f. [2 points] $(\forall n \in \mathbb{N})(n < 2013 \Rightarrow P(n)) \wedge (\forall n \in \mathbb{N})(n \geq 2013 \Rightarrow \neg P(n))$.

9. [5 points] Use induction to prove that for all positive integers n, all of the entries in the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n$$

are $\leq 2n$. (Hint: Find a way to strengthen the inductive hypothesis!)

10. [9 points] In this problem, you are asked to put yourself in the position of a professor, TA, or reader! For each of the induction "proofs" below, say whether you think the proof is correct or incorrect. If you think the proof is incorrect, explain clearly and concisely where the logical error in the proof lies. (If you think the proof is correct, you do not need to give any explanation.) Recall from the notes and from class that simply saying that the claim (or the inductive hypothesis) is false is not a valid explanation.

   10a. [3 points] **Theorem:** For every $n \in \mathbb{N}$, if $n \geq 4$ then $2^n < n!$.

   Proof. The proof will be by induction on n.

   Base case: $2^4 = 16$ and $4! = 24$ and $16 < 24$, so the statement is true for $n = 4$.

   Inductive hypothesis: Assume $n \geq 4$ and $2^n < n!$.

   Inductive step: We have $2^{n+1} = 2 \times 2^n < 2 \times n! < (n+1) \times n! = (n+1)!$ where the first inequality follows by $2^n < n!$ (from the inductive hypothesis) and the second inequality follows by $n \geq 4$. We have thus shown $2^{n+1} < (n+1)!$, and this completes the inductive step. ☐

   10b. [3 points] **Theorem:** For all $x, y, n \in \mathbb{N}$, if $\max(x, y) = n$, then $x \leq y$.

   Proof. The proof will be by induction on n.

Base case: Suppose that $n = 0$. If $\max(x, y) = 0$ and $x, y \in \mathbb{N}$, then $x = 0$ and $y = 0$, hence $x \leq y$.

Inductive hypothesis: Assume that, whenever we have $\max(x, y) = n$, then $x \leq y$ must hold.

Inductive step: We must prove that if $\max(x, y) = n + 1$, then $x \leq y$. Suppose x, y are such that $\max(x, y) = n + 1$. Then it follows that $\max(x - 1, y - 1) = n$, so by the inductive hypothesis, $x - 1 \leq y - 1$. This implies that $x \leq y$, completing the inductive step. $\square$

10c. [3 points] **Theorem:** For all $n \in \mathbb{N}$, $n < 2^n$.

Proof. The proof will be by induction on n.

Base case: $2^0 = 1$ which is greater than 0, so the statement is true for $n = 0$.

Inductive hypothesis: Assume that $n < 2^n$.

Inductive step: We need to show that $n + 1 < 2^{n+1}$. By the inductive hypothesis, we know that $n < 2^n$. Plugging in $n + 1$ in place of n, we get $n + 1 < 2^{n+1}$, which is what we needed to show. This completes the inductive step. $\square$

11. [0 points] **This is an optional challenge problem and is not to be turned in. Give it a try if you're feeling comfortable with the other problems!**

Use simple induction on k to prove that for every positive integer k, the following is true:

> For every real number $r > 0$, there are only finitely many solutions in positive integers to $\frac{1}{n_1} + \cdots + \frac{1}{n_k} = r$.

In other words, there exists some number m (that depends on k and r) such that there are at most m ways of choosing a positive integer n_1, and a (possibly different) positive integer n_2, etc., that satisfy the equation.