Midterm 2

Friday, August 2, 5:10pm–7:10pm

CS 70: Discrete Mathematics and Probability Theory, Summer 2013

Your name ______________________________

Your student ID # ______________________________

Your section # ______________________________

This exam has 5 questions and a total of 100 points.
Do not open the exam until you are told to.
Do not write below this line.

Q1	
Q2	
Q3	
Q4	
Q5	
Total	

1. [30 points] **MULTIPLE CHOICE.** Circle the correct answer. You don't need to show work.

1a. [2 points] If every node in an undirected graph has odd degree, then the number of nodes must be even.

TRUE FALSE

1b. [2 points] An undirected graph with an odd number of edges can never have an eulerian cycle.

TRUE FALSE

1c. [2 points] Suppose an undirected graph has an eulerian cycle that is also a hamiltonian cycle. Then the degree of every node must be exactly 2.

TRUE FALSE

1d. [2 points] Which one of the following is *not* a De Bruijn sequence for $k = 3$?

10001011 10110001 11101000

1e. [2 points] In an n-dimensional hypercube (where $n \geq 2$), for every node there exists a cycle of length 4 containing that node.

TRUE FALSE

1f. [2 points] How many different anagrams of REPETITION are there?

$\frac{10!}{(2!)^3}$ $\binom{10}{2} \times \binom{8}{2} \times \binom{6}{2}$ $10!$ $7! \times 8 \times 9$

1g. [2 points] Suppose you are teaching a course at Berkeley. You have 4 readers, and 50 hours worth of grading to distribute among the readers. In how many ways can this be done? (Assume that each reader gets an integer number of hours, it is acceptable to assign 0 hours to any reader, and the readers don't care what they grade — only how many hours they get.)

4^{49} $\qquad$ 4^{50} $\qquad$ $\binom{53}{49}$ $\qquad$ $\binom{53}{50}$

1h. [2 points] Suppose we throw k balls into n bins with repetition (any number of balls per bin), where $0 < k < n$. Then the number of outcomes is greater if the balls are unlabeled than if they are labeled.

TRUE $\qquad$ FALSE

1i. [2 points] For all events A, B, if $\Pr[A] > 0$ and $\Pr[B] > 0$ and $\Pr[A \mid B] = \Pr[B \mid A]$, then A and B are independent.

TRUE $\qquad$ FALSE

1j. [2 points] In every *uniform* discrete probability space, $\Pr[A \mid B] = \frac{|A \cap B|}{|B|}$ (assuming $\Pr[B] > 0$).

TRUE $\qquad$ FALSE

1k. [2 points] Suppose we condition on some event B. If the posterior probability of an event A is greater than the prior probability of A, then the posterior probability of $\overline{A}$ is greater than the prior probability of $\overline{A}$.

TRUE $\qquad$ FALSE

1l. [2 points] For all events A, B, we have $\Pr[A \cup B] = 1 - \Pr\left[\,\overline{A}\,\right] \Pr\left[\,\overline{B} \mid \overline{A}\,\right]$.

TRUE $\qquad$ FALSE

1m. [2 points] In a random permutation on n elements, the expected number of fixed points equals the expected value of the binomial distribution $B(n, 1/n)$.

TRUE FALSE

1n. [2 points] Suppose X is a random variable that only takes on natural number values. Which one of the following four things is NOT a valid expression for $\mathrm{E}(X)$?

$$\sum_{i=0}^{\infty} i \times \Pr[X = i] \qquad \sum_{\omega \in \Omega} X(\omega) \times \Pr[\omega]$$

$$\sum_{i=1}^{\infty} \Pr[X \geq i] \qquad \sum_{i=0}^{\infty} \mathrm{E}(X_i) \text{ where } X_i \text{ is the indicator for } X = i$$

1o. [2 points] Suppose X and Y are random variables. Which one of the following three things does NOT guarantee that $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$?

X is a constant X, Y are indicators X, Y are independent

2. [15 points] **GRAPHS.**

   2a. [7 points] Consider undirected graphs where we allow multiple edges between pairs of nodes. Prove the following proposition: For every connected graph that has n nodes and does not have an eulerian cycle, it is always possible to add at most $n/2$ edges so that it does have an eulerian cycle.

   2b. [2 points] Does the proposition in part 2a remain true if we forbid multi-edges? Briefly justify your answer.

2c. [6 points] Recall that a *tournament* is a directed graph such that for every pair of distinct nodes v and w, exactly one of (v, w) and (w, v) is an edge. Prove that in every tournament on n nodes, there exists a node with outdegree at least $\frac{n-1}{2}$. (In other words, somebody beats at least half of the other people!)

3. [15 points] **COUNTING.**

You are teaching a class with 100 students. You give them a list of 5 different project suggestions, and you require that each student chooses exactly one of the suggested projects.

3a. [3 points] In how many ways can the students choose their projects?

3b. [3 points] After the students choose, you want to gather the information about how many students chose each project (not caring which student chose which project). How many possibilities are there for this information?

3c. [3 points] Consider a project to be *chosen* if at least one student chooses it. How many different possible sets of chosen projects are there? (Note that it is not possible that none of the projects are chosen, but it is possible that all students choose the same project.)

3d. [3 points] Suppose you insist that each project is chosen by exactly 20 students. In how many ways can the students choose their projects now?

3e. [3 points] Each of the 100 students must give a presentation, and there are 130 different time slots for presentations. No two students can share the same time slot. In how many ways can the students choose time slots?

4. [20 points] **CONDITIONAL PROBABILITY AND INFERENCE.**

You're helping run a medical study which is testing drugs for a particular disease. There are 100 patients with the disease: 50 of them are given drug 1, 30 of them are given drug 2, and 20 of them are given the placebo. If a patient is given drug 1, they are cured with probability 0.6. If given drug 2, they are cured with probability 1.0. If given the placebo, they are cured with probability 0.1.

4a. [7 points] Suppose you select a random patient and observe that they are cured after the study. What is the probability they were given the placebo? Show all your work, and simplify your final answer as much as possible.

4b. [7 points] Suppose instead that you select a random patient, and all you know is that they were *not* given the placebo. What is the probability they will be cured? Show all your work, and simplify your final answer as much as possible.

4c. [6 points] Suppose we have a probability space with a partition $A_1, A_2, \ldots, A_n$ of Ω, and an event B. Suppose we know the value of $\Pr[B \mid A_i]$ for all i, and each of these probabilities is positive. Recall that the Inference Rule allows you to compute the posterior probabilities $\Pr[A_i \mid B]$ from the prior probabilities $\Pr[A_i]$. Find an "Inverse Inference Rule" that allows you to compute the prior probabilities from the posterior probabilities.

5. [20 points] **EXPECTATION.**

5a. [5 points] You roll a 6-sided die and let X be its value (1, 2, 3, 4, 5, or 6). Let $Y = (X \bmod 4)$. Compute $\mathrm{E}(Y)$. Show your work.

5b. [8 points] There are $2n$ people, who are paired up into n "husband-wife pairs". You arrange the people in a line of $2n$ chairs, by picking one of the $(2n)!$ permutations uniformly at random. What is the expected number of husband-wife pairs who are seated next to each other? Show your work.

5c. [2 points] You have two coins, one with heads probability p and one with heads probability q. You flip the p coin n times, then you flip the q coin m times. What is the expected number of heads you see in total?

5d. [5 points] Continuing from (5c), suppose that after flipping the p coin n times, you instead flip the q coin X times where X is the number of heads you saw for the p coin. What is the expected number of heads you see in total? Show your work and simplify your answer as much as possible.