CS 70 FALL 2006 — DISCUSSION #1

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1. Administrivia

(1) Course Information
- The first homework is due September 1st at 4pm in 283 Soda Hall, and is available at `http://www-inst.eecs.berkeley.edu/~cs70/fa06`
- You are encouraged to work on the homework in groups of 3-4, but write up your submission *on your own*. Cite any external sources you use.

(2) Discussion Information
- If you have a clash, it is OK to attend a section different to your enrolled/wait-listed one. Just be sure to show up so that we can 'assign' you somewhere based on the roles taken in sections in the first few weeks.
- Section notes like this one will be posted on the course website.
- Feel free to contact the GSI's via e-mail, or the class staff and students through the newsgroup, if you have a question.
- Remember that 5% of your final grade comes from participation!

2. Warm-up: Can You Spend Exactly Ten Coins?

To help get the problem-solving juices flowing, attempt the following – no special background math is required!

Exercise 1. For your summer vacation you decide to visit the exotic land of Exactavia, where it is customary to pay shop-keepers the *exact amount* due with *exactly ten coins*. The currency of Exactavia consists of 1, 3 and 5 dollar coins. Assume that you have at least ten of each type of coin.

(i) Can you pay \$24? If so, provide the coins you'd use; otherwise explain why it is impossible.
(ii) What if you are asked to pay \$25?
(iii) Can you generalize your procedure to paying amounts a, for $10 \leq a \leq 50$? Which of these amounts can you pay?

□

3. Knights and Knaves: Fun with Propositional Logic

Knights are always truthful while knaves are consistent liars. With this in mind, consider the following conversation between Alice and Bob.

Date: August 30, 2006.

The authors gratefully acknowledge Chris Crutchfield and Amir Kamil for the use of their previous notes, which form part of the basis for this handout.

Alice: "At least one of us is a knight."
Bob: "At least one of us is a knight."

How might we determine whether Alice is a knight or a knave? As we're dealing with the truth of statements (i.e. that "Alice is a knight" and the same for Bob), propositional logic should come to mind! Let's begin by labeling the two propositions we're interested in:

$$\begin{aligned} P &= \text{"Alice is a knight"} \\ Q &= \text{"Bob is a knight"} \end{aligned}$$

Exercise 2. Using the propositional machinery discussed in class, determine exactly what can be deduced from the above facts/statements.

(i) As a first step, write out the *four* distinct statements in terms of P and Q that must be true. (**hint:** what if Alice is/is not a knight, what about Bob?).
(ii) P must be either true or false, as must be Q. Also your four logical propositions must be true. Using a truth table determine which truth assignments to P and Q are consistent with the truth of your four statements. From this deduce what can be said about the truth of P and Q.

□

Using the same steps we can solve many variants of the knights and knaves problem – this is *generalization* in problem solving.

Exercise 3. Repeat the previous exercise after Alice says "At least one of us is a knight" and Bob says "We're both of the same (knight/knave) type."

4. De Morgan's Laws, Quantifiers and Negation

At the end of lecture #1's notes the section 'Quantifiers and Negation' describes how to push a negation passed a quantifier. In particular the following two laws are justified intuitively.

$$\neg\left(\forall x P(x)\right) \equiv \exists x \neg P(x) \tag{4.1}$$

$$\neg\left(\exists x P(x)\right) \equiv \forall x \neg P(x) \tag{4.2}$$

How might these "De Morgan's Laws" be proven formally? In fact in our last class (class #2) we saw that for a *universe* of size 2 these laws translate to the following binary cases.

$$\neg\left(P \wedge Q\right) \equiv \neg P \vee \neg Q \tag{4.3}$$

$$\neg\left(P \vee Q\right) \equiv \neg P \wedge \neg Q \tag{4.4}$$

Following the argument given in lecture, we can appeal to common sense to convince ourselves of the truth of these two equivalences. Say P is the proposition 'Mario is a plumber' and Q represents the logical statement 'Luigi wears green'. Consider Rule (4.4). Intuitively $\neg\left(P \vee Q\right)$ is true if and only if Mario is not a plumber and Luigi is not wearing green. To prove the two (binary) De Morgan

Laws we can also formally test equivalence using truth tables. Let's do so for practice now.

Exercise 4. Prove De Morgan's Laws on two variables.

(i) Prove Rule (4.3), by filling out the following table.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T					
T	F					
F	T					
F	F					

(ii) Use this next table to prove Rule (4.4).

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T					
T	F					
F	T					
F	F					

□

Arguing that a universally (existentially) quantified statement is equivalent to a big possibly even infinite conjunction (disjunction), the common-sense argument leads us to the general De Morgan's Laws.

Exercise 5. How might you 'formally' prove the two De Morgan's Laws for universes of arbitrary finite size?

Next week we'll learn about induction in lectures. One application of induction is to formally proving the laws on countably in/finite universes. Extensions of induction are capable of proving the laws for uncountable universes.

5. Quantifier Practice

Consider the false statement "For each x in $\mathbb{R}$, $x^2 \geq x$" (consider $0 < x < 1$). What is the negation of this statement? Is it "For each x in $\mathbb{R}$, $x^2 < x$"? No, because this statement is still false (e.g. consider $x > 1$). So what is going wrong here?

Let $P(x)$ be the proposition "$x^2 \geq x$" with x taken from the universe of real numbers $\mathbb{R}$. Then our original statement is succinctly written as $\forall x, P(x)$. From our previous discussion on De Morgan's Laws and their generalization, we can apply Rule (4.1) to get $\neg\forall x, P(x) \equiv \exists x, \neg P(x)$ or "There exists a real x for which $x^2 < x$."

We can chain together quantifiers in any manner we please: $\forall x, \exists y, \forall z, P(x,y,z)$ and negate it using the same rules discussed above. By applying the rules in sequence, we get that

$$\neg(\forall x, \exists y, \forall z, P(x,y,z))$$
$$\exists x, \neg(\exists y, \forall z, P(x,y,z))$$
$$\exists x, \forall y, \neg(\forall z, P(x,y,z))$$
$$\exists x, \forall y, \exists z, \neg P(x,y,z)$$

The $\neg$ "bubbles down", flipping quantifiers as it goes. The following problem comes from Question 14 in the Mathematics Subject GRE Sample Test:

Exercise 6. Let $\mathbb{R}$ be the set of real numbers and let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. The negation of the statement

"For each s in $\mathbb{R}$, there exists an r in $\mathbb{R}$ such that if $f(r) > 0$, then $g(s) > 0$."

is which of the following?

(A) For each s in $\mathbb{R}$, there exists an r in $\mathbb{R}$ such that $f(r) \leq 0$ and $g(s) > 0$.
(B) There exists an s in $\mathbb{R}$ such that for each r in $\mathbb{R}$, $f(r) \leq 0$ and $g(s) \leq 0$.
(C) There exists an s in $\mathbb{R}$ such that for each r in $\mathbb{R}$, $f(r) \leq 0$ and $g(s) > 0$.
(D) There exists an s in $\mathbb{R}$ such that for each r in $\mathbb{R}$, $f(r) > 0$ and $g(s) \leq 0$.
(E) For each s in $\mathbb{R}$, there exists an r in $\mathbb{R}$ such that $f(r) \leq 0$ and $g(s) \leq 0$.

Use the tools covered above. (**hint:** what happens when you negate an implication? Try rewriting the statements in propositional logic, e.g. replacing $f(r) > 0$ with $P(r)$ and $g(s) > 0$ with $Q(s)$). $\square$