Problem Set 2

1. Exclusive OR The exclusive OR (written as XOR or $\oplus$) is just what it sounds like: $P \oplus Q$ is true when exactly one of P, Q is true.

   Show, using a truth table, that $P \oplus Q$ is equivalent to $(P \vee Q) \wedge \neg(P \wedge Q)$.

2. Which of the following conditional sentences is true?

   (a) If triangles have three sides, then squares have four sides.
   (b) If a hexagon has six sides, then the moon is made of cheese.
   (c) If $7 + 6 = 14$, then $5 + 5 = 10$.
   (d) If $5 < 2$, then $10 < 7$.

   Write the converse and contrapositive of each of the conditional sentences listed above.

3. Translate the following English sentence into a symbolic sentence with quantifiers (use the predicate fool(x,t) to denote the fact that you fool person x at some time t): "You can fool some of the people all the time and all of the people some of the time, but you cannot fool all of the people all of the time. (Interpret 'some' as 'at least one').

   Write an english sentence that expresses the negation of the above sentence.

4. Prove that if a^2 is even then a is even. What style of proof did you use?

5. Suppose $P(n)$ is a predicate on the natural numbers, and suppose $\forall k \in \mathcal{N}\ P(k) \implies P(k+2)$.

   For each of the following assertions below, state whether (A) it must always hold, or (N) it can never hold, or (C) it can hold but need not always.

   Give a very brief (one or two sentence) justification for your answers. The domain of all quantifiers is the natural numbers.

   (a) $\forall n \geq 0 \ \ P(n)$

   (b) $P(0) \implies \forall n\ P(n+2)$

   (c) $P(0) \implies \forall n\ P(n+1)$

   (d) $\forall n \ \ \neg P(n)$

   (e) $(\forall n \leq 10 \ \ P(n)) \wedge (\forall n > 10 \ \ \neg P(n))$

   (f) $(\forall n \leq 10 \ \ \neg P(n)) \wedge (\forall n > 10 \ \ P(n))$

(g) $P(0) \wedge P(1) \implies \forall n\ P(n)$

(h) $P(n) \implies [\exists m > n\ \ P(m)]$

(i) $\neg P(m) \vee \neg P(m+1) \vee (\forall n \geq m\ \ P(n))$

6. (a) Prove by induction that $\prod_{i=2}^{n}(1 - 1/i) = 1/n$.

   (b) Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ for all integers $n > 0$.

   (c) *Tower of Brahma* This game is also known as the Towers of Hanoi or the End of the World Puzzle. It was invented by the French mathematician, Edouard Lucas, in 1883. Accompanying the puzzle is a story:

   In the great temple at Benares beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and Night unceasingly, the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When all the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

   Prove by induction the exact number of moves required to carry out this task in general, if there are n disks on the original needle. Assuming that the priests can move a disk each second, roughly how many centuries does the prophecy predict before the destruction of the World?

7. *Proofs to Grade* Assign a grade of A (excellent) if the claim and proof are correct, F (failure) if the claim is incorrect, if the main idea in the proof is incorrect, or if most of the statements in it are incorrect. Assign a grade of C (partial credit) for a proof that is largely correct, but contains one or two incorrect statements or justifications. Whenever a proof is incorrect, explain your grade. Say what is incorrect and why.

   (a) Suppose m is an integer

   Claim: If m^2 is odd then m is odd.

   Proof: Assume m is odd. Then $m = 2k + 1$ for some integer k. Therefore $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. Therefore if m^2 is odd, then m is odd.

   (b) Suppose m is an integer

   Claim: If m^2 is odd then m is odd.

   Proof: Assume that m^2 is not odd. Then m^2 is even, and $m^2 = 2k$ for some integer k. Thus $2k$ is a perfect square; that is, $\sqrt{2k}$ is an integer. If $\sqrt{2k}$ is odd, then $\sqrt{2k} = 2n + 1$ for some integer n, which means $m^2 = 2k = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$. Thus m^2 is odd, contrary to our assumption. Therefore $\sqrt{2k} = m$ must be even. Thus if m^2 is not odd, then m is not odd. Hence if m^2 is odd, then m is odd.

   (c) Suppose t is a real number.

   Claim: If t is irrational, then $5t$ is irrational.

   Proof: Suppose $5t$ is rational. Then $5t = p/q$ where p and q are integers and $q \neq 0$. Therefore $t = p/5q$ where p and $5q$ are integers and $5q \neq 0$, so t is rational. Therefore if t is irrational, then $5t$ is irrational.

(d) Suppose x and y are integers.

Claim: If x and y are even, then $x + y$ is even.

Proof: Suppose x and y are even but $x + y$ is odd. Then, for some integer k, $x + y = 2k + 1$. Therefore $x + y + (-2)k = 1$. The left side of the equation is even because it is the sum of even numbers. However the right side, 1, is odd. Since and even cannot equal an odd, we have a contradiction. Therefore $x + y$ is even. ♠

(e) **Claim:** For every $n \in \mathcal{N}$, $n^2 + n$ is odd. **Proof:** The number $n = 1$ is odd. Suppose $k \in \mathcal{N}$ and $k^2 + k$ is odd. Then,

$$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + (2k + 2)$$

is the sum of an odd and an even integer. Therefore, $(k+1)^2 + (k+1)$ is odd. By the Principle of Mathematical Induction, the property that $n^2 + n$ is odd is true for all natural numbers n. ♠