CS61B Lecture #16: Complexity
What Are the Questions?

• Cost is a principal concern throughout engineering:
  “An engineer is someone who can do for a dime what any fool can do for a dollar.”

• Cost can mean
  - Operational cost (for programs, time to run, space requirements).
  - Development costs: How much engineering time? When delivered?
  - Maintenance costs: Upgrades, bug fixes.
  - Costs of failure: How robust? How safe?

• Is this program fast enough? Depends on:
  - For what purpose;
  - For what input data.

• How much space (memory, disk space)?
  - Again depends on what input data.

• How will it scale, as input gets big?
Problem: Scan a text corpus (say $10^9$ bytes or so), and find and print the 20 most frequently used words, together with counts of how often they occur.

- Solution 1 (Knuth): Heavy-Duty data structures
  - Hash Trie implementation, randomized placement, pointers galore, several pages long.

- Solution 2 (Doug McIlroy): UNIX shell script:
  
  tr -c -s '[[:alpha:]]' '[\n*]' < FILE | 
  sort | 
  uniq -c | 
  sort -n -r -k 1,1 | 
  sed 20q

- Which is better?
  - #1 is much faster,
  - but #2 took 5 minutes to write and processes 1GB in $\approx 256$ sec.
  - I pick #2.

- In very many cases, almost anything will do: Keep It Simple.
Cost Measures (Time)

- **Wall-clock or execution time**
  - You can do this at home:
    
    ```
    time java FindPrimes 1000
    ```
  - Advantages: easy to measure, meaning is obvious.
  - Appropriate where time is critical (real-time systems, e.g.).
  - Disadvantages: applies only to specific data set, compiler, machine, etc.

- **Dynamic statement counts** of # of times statements are executed:
  - Advantages: more general (not sensitive to speed of machine).
  - Disadvantages: doesn’t tell you actual time, still applies only to specific data sets.

- **Symbolic execution times:**
  - That is, *formulas* for execution times as functions of input size.
  - Advantages: applies to all inputs, makes scaling clear.
  - Disadvantage: practical formula must be approximate, may tell very little about actual time.
Asymptotic Cost

- Symbolic execution time lets us see *shape* of the cost function.
- Since we are approximating anyway, pointless to be precise about certain things:
  - *Behavior on small inputs*:
    - Can always pre-calculate some results.
    - Times for small inputs not usually important.
    - Often more interested in *asymptotic behavior* as input size becomes very large.
  - *Constant factors* (as in “off by factor of 2”):
    - Just changing machines causes constant-factor change.
- How to abstract away from (i.e., ignore) these things?
Handy Tool: Order Notation

• Idea: Don’t try to produce specific functions that specify size, but rather families of functions with similarly behaved magnitudes.

• Then say something like “$f$ is bounded by $g$ if it is in $g$’s family.”

• For any function $g(x)$, the functions $2g(x)$, $0.5g(x)$, or for any $K > 0$, $K \cdot g(x)$, all have the same “shape”. So put all of them into $g$’s family.

• Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant $M$) has $g$’s shape “except for small values.” So put all of these in $g$’s family.

• For upper limits, throw in all functions whose absolute value is everywhere $\leq$ some member of $g$’s family. Call this set $O(g)$ or $O(g(n))$.

• Or, for lower limits, throw in all functions whose absolute value is everywhere $\geq$ some member of $g$’s family. Call this set $\Omega(g)$.

• Finally, define $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions bracketed in magnitude by two members of $g$’s family.
Goal: Specify bounding from above.

Here, $f(x) \leq 2g(x)$ as long as $x > 1$,

So $f(x)$ is in $g$'s “bounded-above family,” written

$$f(x) \in O(g(x)),$$

...even though (in this case) $f(x) > g(x)$ everywhere.
Big Omega

- **Goal:** Specify bounding from below:

\[ g(x) \]

- Here, \( f'(x) \geq \frac{1}{2}g(x) \) as long as \( x > 1 \).
- So \( f'(x) \) is in \( g \)'s “bounded-below family,” written

\[ f'(x) \in \Omega(g(x)) \],

- ...even though \( f(x) < g(x) \) everywhere.
Big Theta

- In the two previous slides, we not only have \( f(x) \in O(g(x)) \) and 
  \( f'(x) \in \Omega(g(x)) \),...
- ...but also \( f(x) \in \Omega(g(x)) \) and \( f'(x) \in O(g(x)) \).
- We can summarize this all by saying \( f(x) \in \Theta(g(x)) \) and \( f'(x) \in \Theta(g(x)) \).
Aside: Various Mathematical Pedantry

- Technically, if I am going to talk about $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ as sets of functions, I really should write, for example,
  \[ f \in O(g) \text{ instead of } f(x) \in O(g(x)) \]

- In effect, $f(x) \in O(g(x))$ is short for $\lambda x. f(x) \in O(\lambda x. g(x))$.

- The standard notation outside this course, in fact, is $f(x) = O(g(x))$, but personally, I think that’s a serious abuse of notation.
How We Use Order Notation

• Elsewhere in mathematics, you’ll see $O(\ldots)$, etc., used generally to specify bounds on functions.

• For example,

$$\pi(N) = \Theta\left(\frac{N}{\ln N}\right)$$

which I would prefer to write

$$\pi(N) \in \Theta\left(\frac{N}{\ln N}\right)$$

(Here, $\pi(N)$ is the number of primes less than or equal to $N$.)

• Also, you’ll see things like

$$f(x) = x^3 + x^2 + O(x) \quad \text{(or } f(x) \in x^3 + x^2 + O(x)),$$

meaning that $f(x) = x^3 + x^2 + g(x)$ where $g(x) \in O(x)$.

• For our purposes, the functions we will be bounding will be cost functions: functions that measure the amount of execution time or the amount of space required by a program or algorithm.
Why It Matters

• Computer scientists often talk as if constant factors didn’t matter at all, only the difference of \( \Theta(N) \) vs. \( \Theta(N^2) \).

• In reality they do matter, but at some point, constants always get swamped.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 16 \lg n )</th>
<th>( \sqrt{n} )</th>
<th>( n )</th>
<th>( n \lg n )</th>
<th>( n^2 )</th>
<th>( n^3 )</th>
<th>( 2^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>1.4</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>64</td>
<td>16</td>
</tr>
<tr>
<td>8</td>
<td>48</td>
<td>2.8</td>
<td>8</td>
<td>24</td>
<td>64</td>
<td>512</td>
<td>256</td>
</tr>
<tr>
<td>16</td>
<td>64</td>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>4,096</td>
<td>65,636</td>
</tr>
<tr>
<td>32</td>
<td>80</td>
<td>5.7</td>
<td>32</td>
<td>160</td>
<td>1024</td>
<td>32,768</td>
<td>4.2 \times 10^9</td>
</tr>
<tr>
<td>64</td>
<td>96</td>
<td>8</td>
<td>64</td>
<td>384</td>
<td>4,096</td>
<td>262,144</td>
<td>1.8 \times 10^{19}</td>
</tr>
<tr>
<td>128</td>
<td>112</td>
<td>11</td>
<td>128</td>
<td>896</td>
<td>16,384</td>
<td>2.1 \times 10^9</td>
<td>3.4 \times 10^{38}</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>1,024</td>
<td>160</td>
<td>32</td>
<td>1,024</td>
<td>10,240</td>
<td>1.0 \times 10^6</td>
<td>1.1 \times 10^9</td>
<td>1.8 \times 10^{308}</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( 2^{20} )</td>
<td>320</td>
<td>1024</td>
<td>1.0 \times 10^6</td>
<td>2.1 \times 10^7</td>
<td>1.1 \times 10^{12}</td>
<td>1.2 \times 10^{18}</td>
<td>6.7 \times 10^{315,652}</td>
</tr>
</tbody>
</table>

• For example: replace column \( n^2 \) with \( 10^6 \cdot n^2 \) and it still becomes dominated by \( 2^n \).
Some Intuition on Meaning of Growth

- How big a problem can you solve in a given time?
- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size $N$.
- Entries show the size of problem that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.
- $N =$ problem size.

<table>
<thead>
<tr>
<th>Time ($\mu$sec) for problem size $N$</th>
<th>1 second</th>
<th>Max $N$ Possible in 1 hour</th>
<th>Max $N$ Possible in 1 month</th>
<th>Max $N$ Possible in 1 century</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lg N$</td>
<td>10$^{300000}$</td>
<td>10$^{10000000000}$</td>
<td>10$^{8 \cdot 10^{11}}$</td>
<td>10$^{10^{14}}$</td>
</tr>
<tr>
<td>$N$</td>
<td>10$^6$</td>
<td>3.6 $\cdot$ 10$^9$</td>
<td>2.7 $\cdot$ 10$^{12}$</td>
<td>3.2 $\cdot$ 10$^{15}$</td>
</tr>
<tr>
<td>$N \lg N$</td>
<td>63000</td>
<td>1.3 $\cdot$ 10$^8$</td>
<td>7.4 $\cdot$ 10$^{10}$</td>
<td>6.9 $\cdot$ 10$^{13}$</td>
</tr>
<tr>
<td>$N^2$</td>
<td>1000</td>
<td>60000</td>
<td>1.6 $\cdot$ 10$^6$</td>
<td>5.6 $\cdot$ 10$^7$</td>
</tr>
<tr>
<td>$N^3$</td>
<td>100</td>
<td>1500</td>
<td>14000</td>
<td>150000</td>
</tr>
<tr>
<td>$2^N$</td>
<td>20</td>
<td>32</td>
<td>41</td>
<td>51</td>
</tr>
</tbody>
</table>
Using the Notation

- Can use this order notation for any kind of real-valued function.
- We will use them to describe cost functions. Example:

```java
/** Find position of X in list L, or -1 if not found. */
int find(List L, Object X) {
    int c;
    for (c = 0; L != null; L = L.next, c += 1)
        if (X.equals(L.head)) return c;
    return -1;
}
```

- Choose representative operation: number of .equals tests.
- If \( N \) is length of \( L \), then loop does at most \( N \) tests: worst-case time is \( N \) tests.
- In fact, total \# of instructions executed is roughly proportional to \( N \) in the worst case, so can also say worst-case time is \( O(N) \), regardless of units used to measure.
- Use \( N > M \) provision (in defn. of \( O(\cdot) \)) to ignore empty list.
Be Careful

• It’s also true that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.

• The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does not mean that the loop always takes time $N$, or even $K \cdot N$ for some $K$.

• Instead, we are just saying something about the function that maps $N$ into the largest possible time required to process any array of length $N$.

• To say as much as possible about our worst-case time, we should try to give a $\Theta$ bound: in this case, we can: $\Theta(N)$.

• But again, that still tells us nothing about best-case time, which happens when we find $X$ at the beginning of the loop. Best-case time is $\Theta(1)$. 
Effect of Nested Loops

• Nested loops often lead to polynomial bounds:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = 0; j < A.length; j += 1)
        if (i != j && A[i] == A[j])
            return true;
    return false;
```

• Clearly, time is $O(N^2)$, where $N = A.length$. **Worst-case time is $\Theta(N^2)$**.
Constant Factor Speed-Up

- Previous loop is inefficient. This one is considerably faster:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = i+1; j < A.length; j += 1)
        if (A[i] == A[j]) return true;
return false;
```

- Now worst-case time is proportional to

\[ N - 1 + N - 2 + \ldots + 1 = \frac{N(N - 1)}{2}, \]

which is a constant-factor improvement.

- But still,

\[ \frac{N(N - 1)}{2} \in \Theta(N^2). \]

so the asymptotic time is unchanged by the constant-factor speed-up.
Recursion and Recurrences: Fast Growth

• Silly example of recursion. In the worst case, both recursive calls happen:

```java
/** True iff X is a substring of S */
boolean occurs(String S, String X) {
    if (S.equals(X)) return true;
    if (S.length() <= X.length()) return false;
    return
        occurs(S.substring(1), X) ||
        occurs(S.substring(0, S.length()-1), X);
}
```

• Define $C(N)$ to be the worst-case cost of $\text{occurs}(S,X)$ for $S$ of length $N$, $X$ of fixed size $N_0$, measured in # of calls to $\text{occurs}$. Then

$$C(N) = \begin{cases} 1, & \text{if } N \leq N_0, \\
2C(N - 1) + 1 & \text{if } N > N_0
\end{cases}$$

• So $C(N)$ grows exponentially:

$$C(N) = 2C(N - 1) + 1 = 2(2C(N - 2) + 1) + 1 = \ldots = 2\underbrace{\cdots 2 \cdot 1 + 1}_{N-N_0} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$$
/** True X iff is an element of S[L .. U]. Assumes
 * S in ascending order, 0 <= L <= U-1 < S.length. */
boolean isIn(String X, String[] S, int L, int U) {
    if (L > U) return false;
    int M = (L+U)/2;
    int direct = X.compareTo(S[M]);
    if (direct < 0) return isIn(X, S, L, M-1);
    else if (direct > 0) return isIn(X, S, M+1, U);
    else return true;
}

• Here, worst-case time, \( C(D) \), (as measured by # of calls to .compareTo),
depends on size \( D = U - L + 1 \).

• We eliminate \( S[M] \) from consideration each time and look at half the
rest. Assume \( D = 2^k - 1 \) for simplicity, so:

\[
C(D) = \begin{cases} 
0, & \text{if } D \leq 0, \\
1 + C(((D - 1)/2), & \text{if } D > 0. 
\end{cases}
\]

\[
= 1 + 1 + \ldots + 1 + 0 \\
= k = \lg(D + 1) \in \Theta(\lg D)
\]
Another Typical Pattern: Merge Sort

List sort(List L) {
    if (L.length() < 2) return L;
    Split L into L0 and L1 of about equal size;
    L0 = sort(L0); L1 = sort(L1);
    return Merge of L0 and L1
}

 Merge ("combine into a single ordered list") takes time proportional to size of its result.

- Assuming that size of L is $N = 2^k$, worst-case cost function, $C(N)$, counting just merge time (which is proportional to # items merged):

$$C(N) = \begin{cases} 
0, & \text{if } N < 2; \\
2C(N/2) + N, & \text{if } N \geq 2.
\end{cases}$$

$$= 2(2C(N/4) + N/2) + N$$
$$= 4C(N/4) + N + N$$
$$= 8C(N/8) + N + N + N$$
$$= N \cdot 0 + \underbrace{N + N + \ldots + N}_{k = \log N}$$
$$= N \log N$$

- In general, can say it's $\Theta(N \log N)$ for arbitrary $N$ (not just $2^k$).