CS61B Lecture #37

- **Today:** Minimum spanning trees, recursive graph algorithms, union-find.

Minimum Spanning Trees

- **Problem:** Given a set of places and distances between them (assume always positive), find a set of connecting roads of minimum total length that allows travel between any two.
- The routes you get will not necessarily be shortest paths.
- Easy to see that such a set of connecting roads and places must form a tree, because removing one road in a cycle still allows all to be reached.

Minimum Spanning Trees by Prim's Algorithm

- Idea is to grow a tree starting from an arbitrary node.
- At each step, add the shortest edge connecting some node already in the tree to one that isn't yet.
- Why must this work?

```
PriorityQueue fringe;
For each node v { v.dist() = ∞; v.parent() = null; }
Choose an arbitrary starting node, s;
s.dist() = 0;
fringe = priority queue ordered by smallest .dist();
add all vertices to fringe;
while (! fringe.isEmpty()) {
  Vertex v = fringe.removeFirst ();

  For each edge (v,w) {
    if (w ∈ fringe && weight(v,w) < w.dist())
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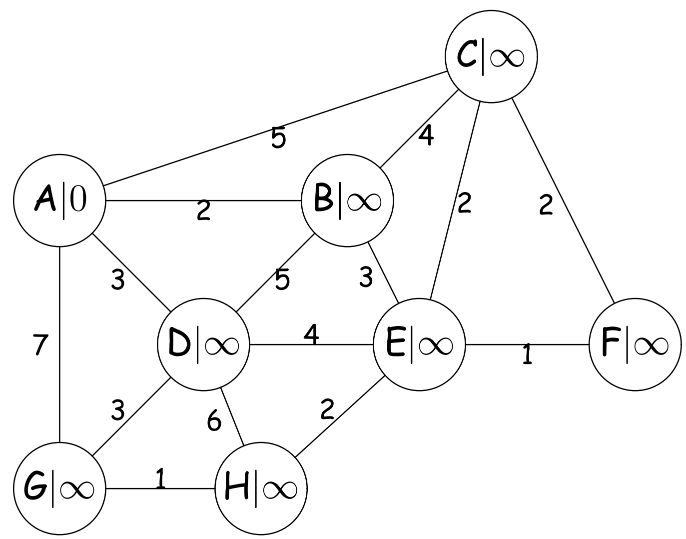

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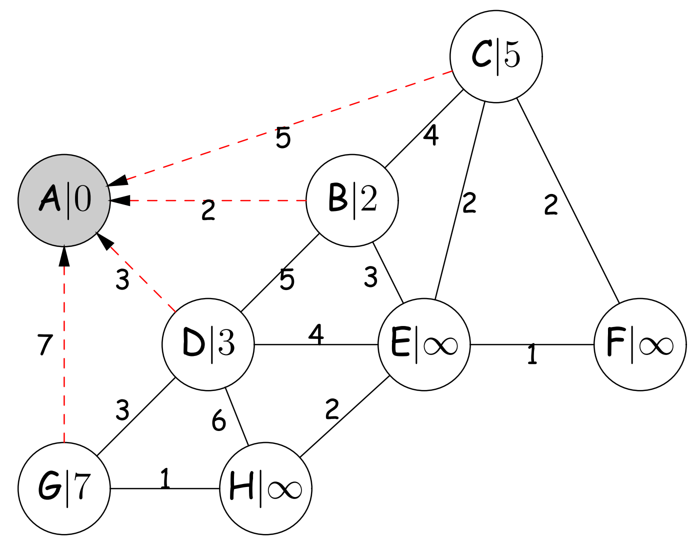

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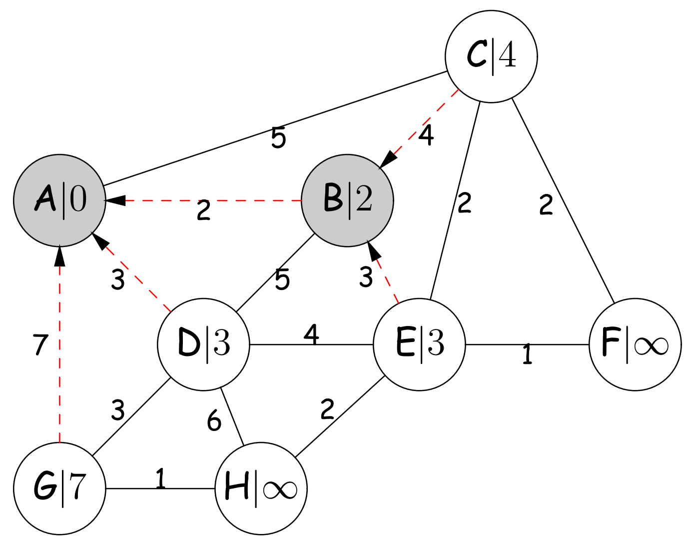

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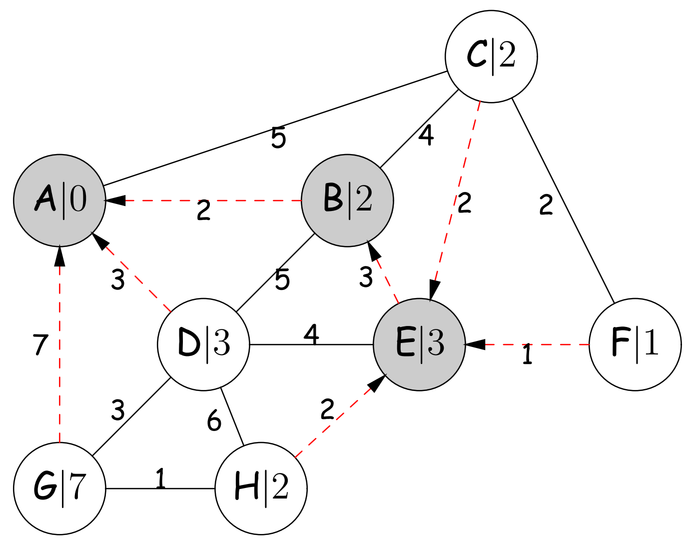

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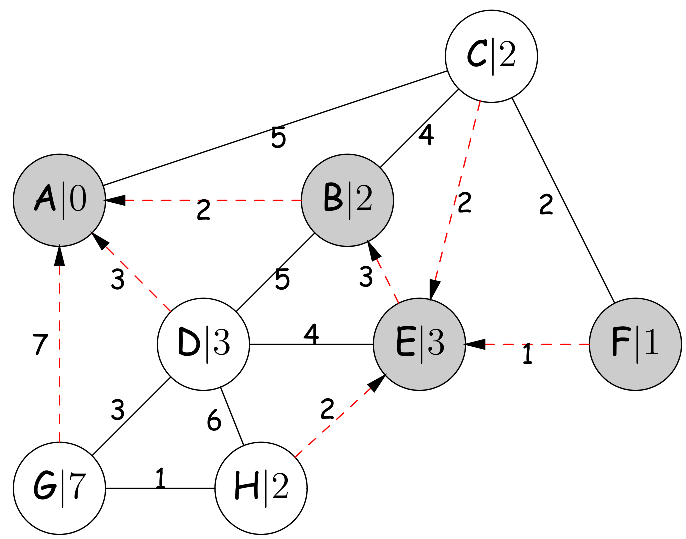

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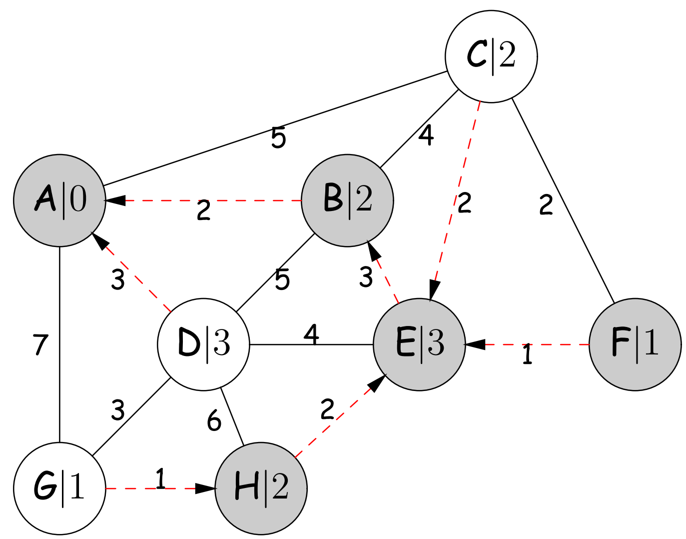

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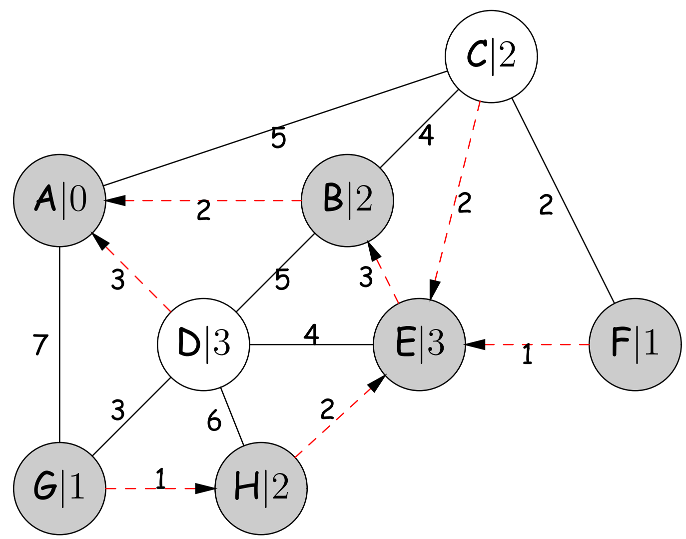

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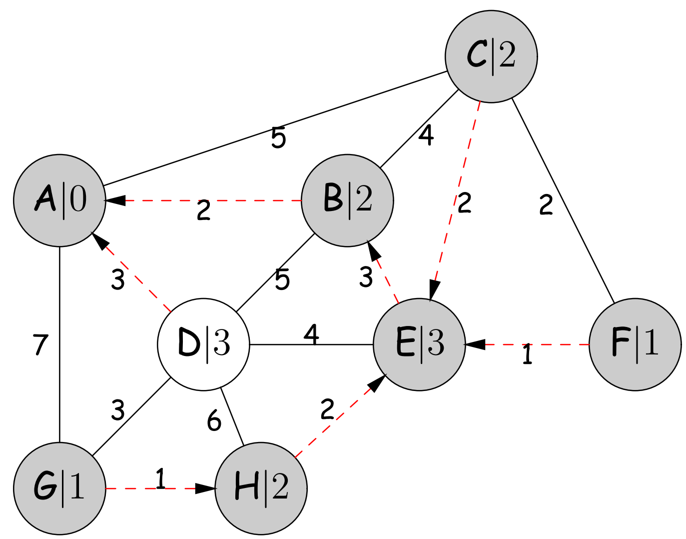

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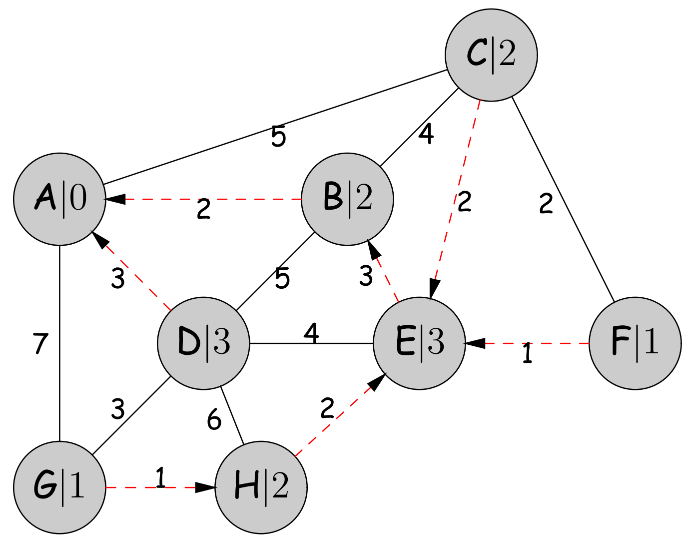


Minimum Spanning Trees by Kruskal's Algorithm

- Observation: the shortest edge in a graph can always be part of a minimum spanning tree.
- In fact, if we have a bunch of subtrees of a MST, then the shortest edge that connects two of them can be part of a MST, combining the two subtrees into a bigger one.
- So,...

```
Create one (trivial) subtree for each node in the graph;
MST = {};

for each edge (v,w), in increasing order of weight {
   if ( (v,w) connects two different subtrees ) {
      Add (v,w) to MST;
      Combine the two subtrees into one;
   }
}
```

Recursive Depth-First Traversal

- Previously, we saw an iterative way to do depth-first traversal of a graph from a particular node.
- We are often interested in traversing all nodes of a graph, so we can repeat the procedure as long as there are unmarked nodes.
- Recursive solution is also simple:

```
void traverse (Graph G) {
   for (v ∈ nodes of G) {
      traverse (G, v);
}

void traverse (Graph G, Node v) {
   if (v is unmarked) {
     mark (v);
     visit v;
     for (Edge (v, w) ∈ G)
       traverse (G, w);
   }
}
```

Another Take on Topological Sort

- Observation: if we do a depth-first traversal on a DAG whose edges are reversed, and execute the recursive `traverse` procedure, we finish executing `traverse(G,v)` in proper topologically sorted order.

```
void topologicalSort (Graph G) {
   for (v ∈ nodes of G) {
      traverse (G, v);
}

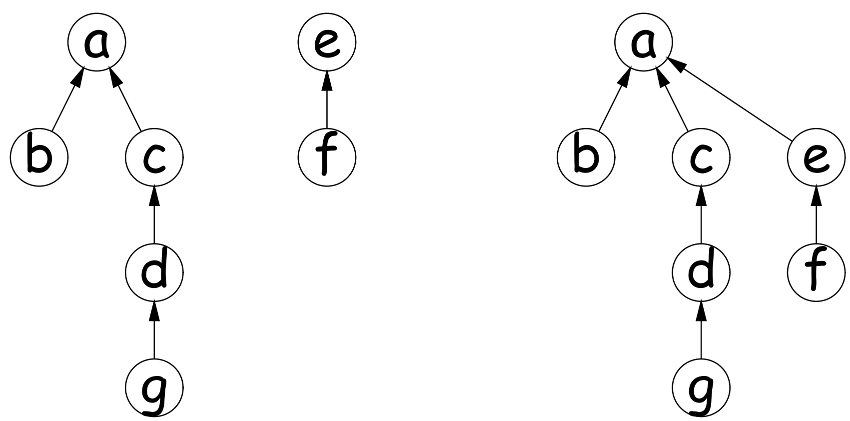
void traverse (Graph G, Node v) {
   if (v is unmarked) {
     mark (v);
     for (Edge (w, v) ∈ G)
       traverse (G, w);
     add v to the result list;
   }
}
```

Union Find

- Kruskal's algorithm required that we have a set of sets of nodes with two operations:
  - *Find* which of the sets a given node belongs to.
  - Replace two sets with their *union*, reassigning all the nodes in the two original sets to this union.
- Obvious thing to do is to store a set number in each node, making finds fast.
- Union requires changing the set number in one of the two sets being merged; the smaller is better choice.
- This means an individual union can take $\Theta(N)$ time.
- Can union be fast?

A Clever Trick

- Let's choose to represent a set of nodes by *one* arbitrary representative node in that set.
- Let every node contain a pointer to another node in the same set.
- Arrange for each pointer to represent the *parent* of a node in a tree that has the representative node as its root.
- To find what set a node is in, follow parent pointers.
- To union two such trees, make one root point to the other (choose the root of the higher tree as the union representative).



Path Compression

- This makes unioning really fast, but the find operation potentially slow ($\Omega(\lg N)$).
- So use the following trick: whenever we do a *find* operation, *compress* the path to the root, so that subsequent finds will be faster.
- That is, make each of the nodes in the path point directly to the root.
- Now union is very fast, and sequence of unions and finds each have very, *very* nearly constant amortized time.
- Example: find 'g' in last tree (result of compression on right):