CS 61A Lecture 11

Announcements

Box-and-Pointer Notation

The Closure Property of Data Types

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  The result of combination can itself be combined using the same method

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Lists can contain lists as elements (in addition to anything else)

Box-and-Pointer Notation in Environment Diagrams

Interactive Diagram

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Lists are represented as a row of index-labeled adjacent boxes, one per element

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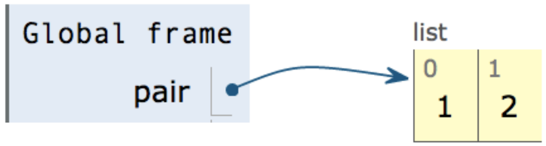


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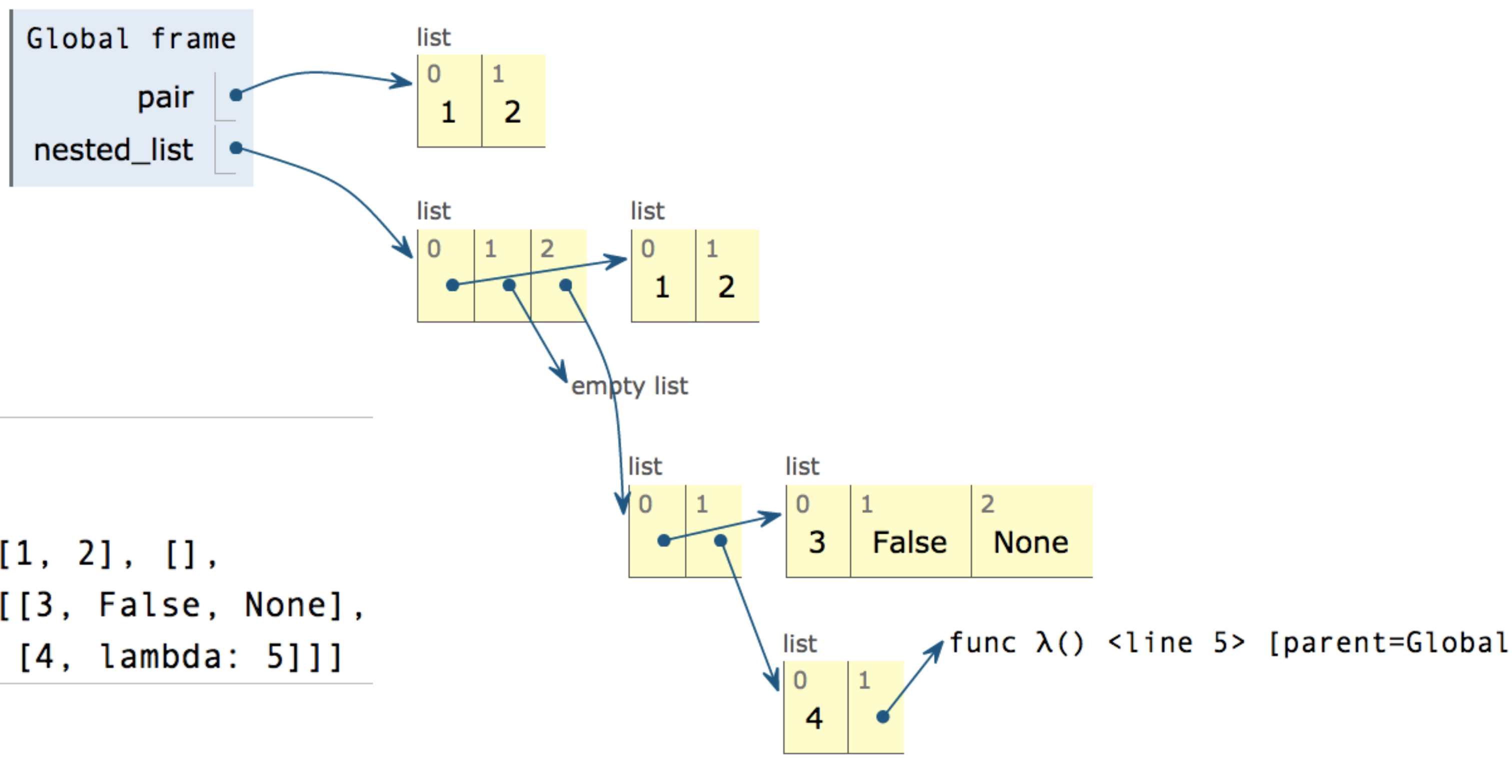


```
pair = [1, 2]

nested_list = [[1, 2], [],
               [[3, False, None],
                [4, lambda: 5]]]
```

Interactive Diagram

Sequence Operations

Membership & Slicing

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Python sequences have operators for membership and slicing
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Slicing creates a new object

Membership & Slicing

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middle = digits[1:3]
end =    digits[2:]
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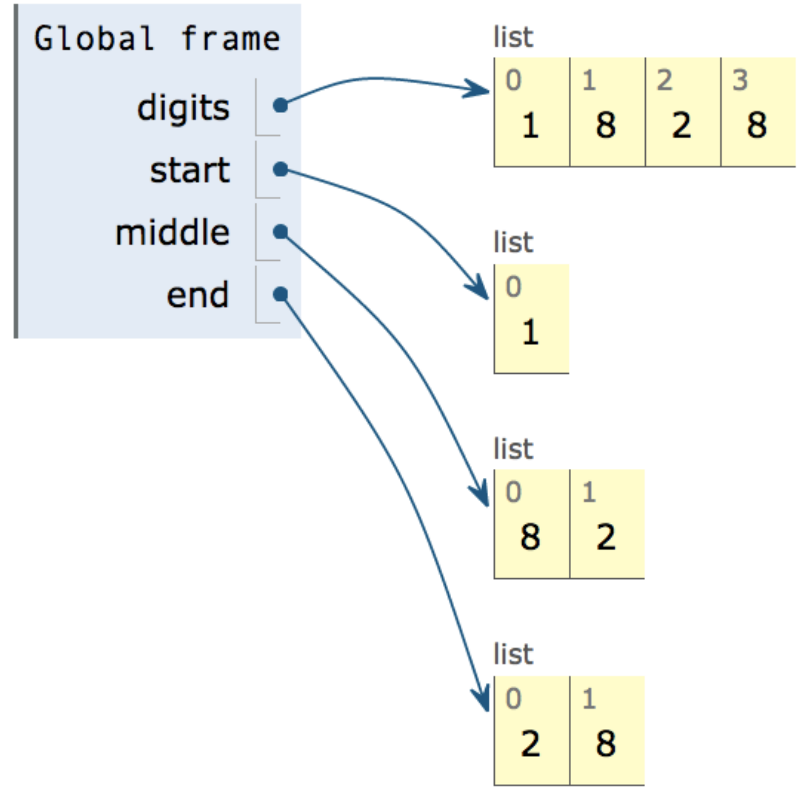

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Trees

Tree Abstraction

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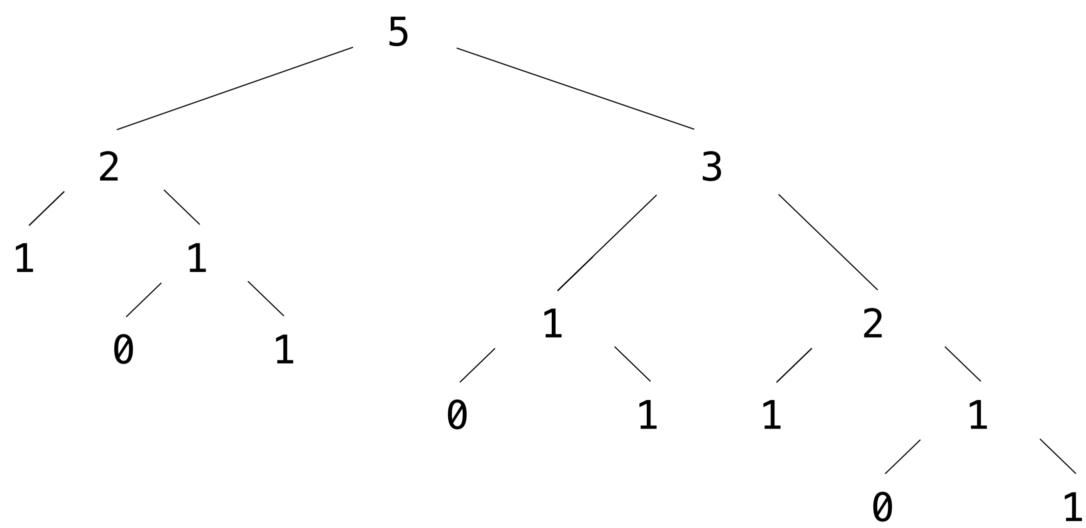

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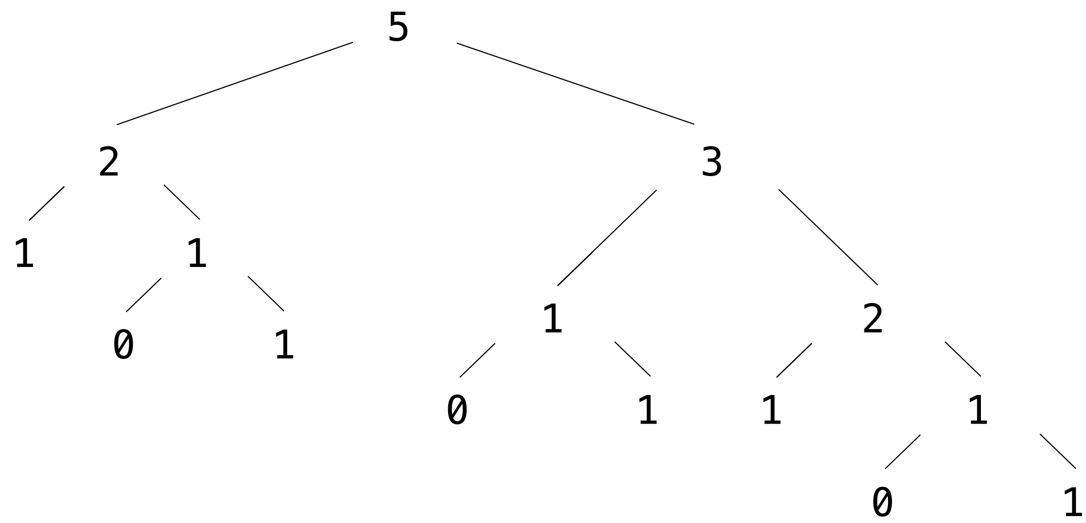


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A tree has a root value and a sequence of branches; each branch is a tree
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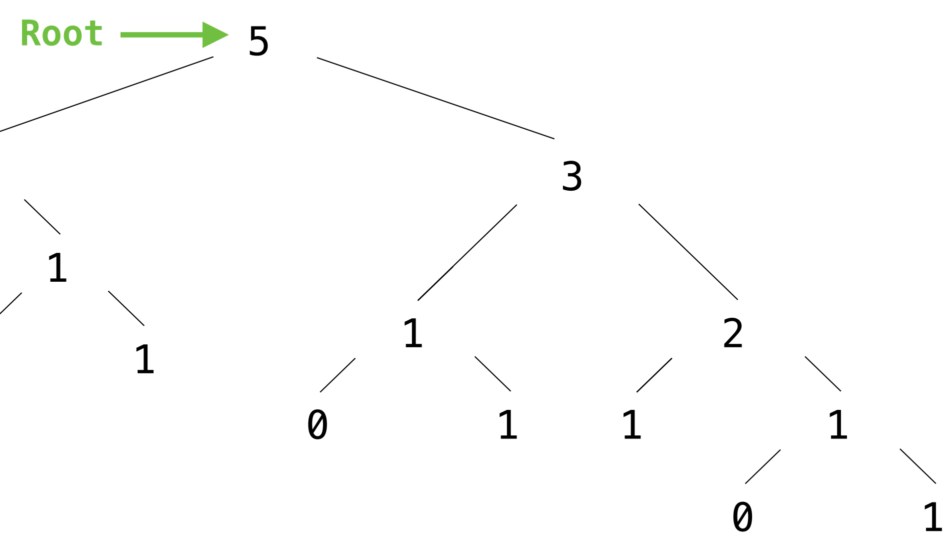

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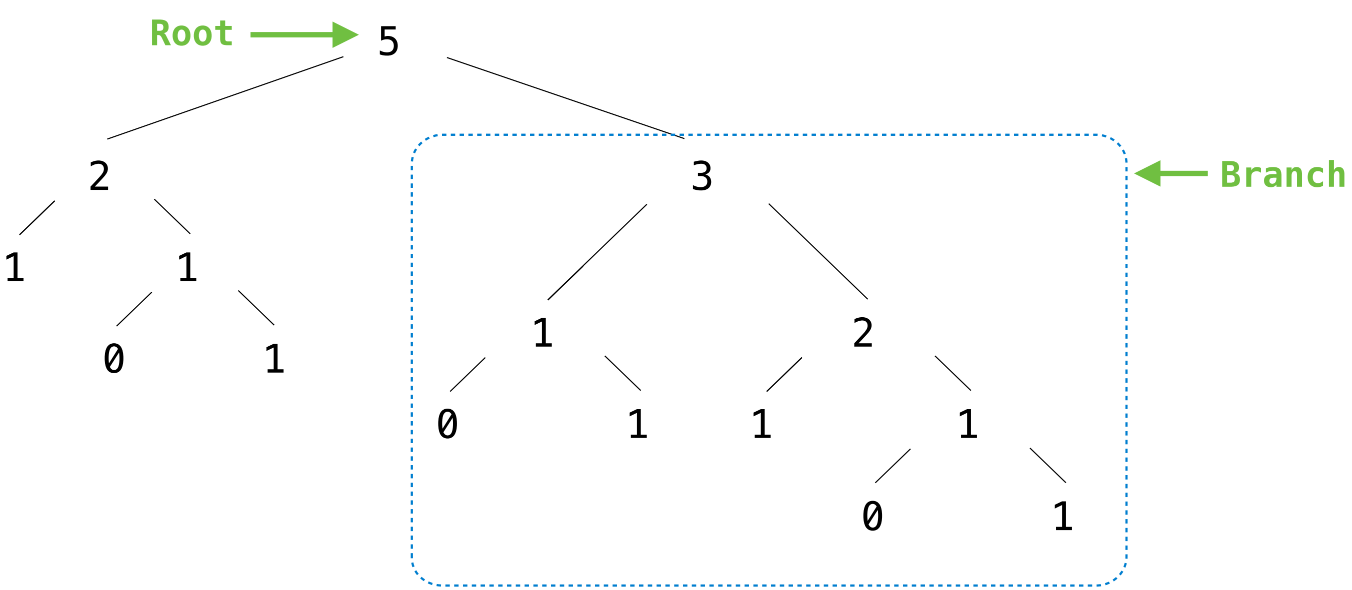

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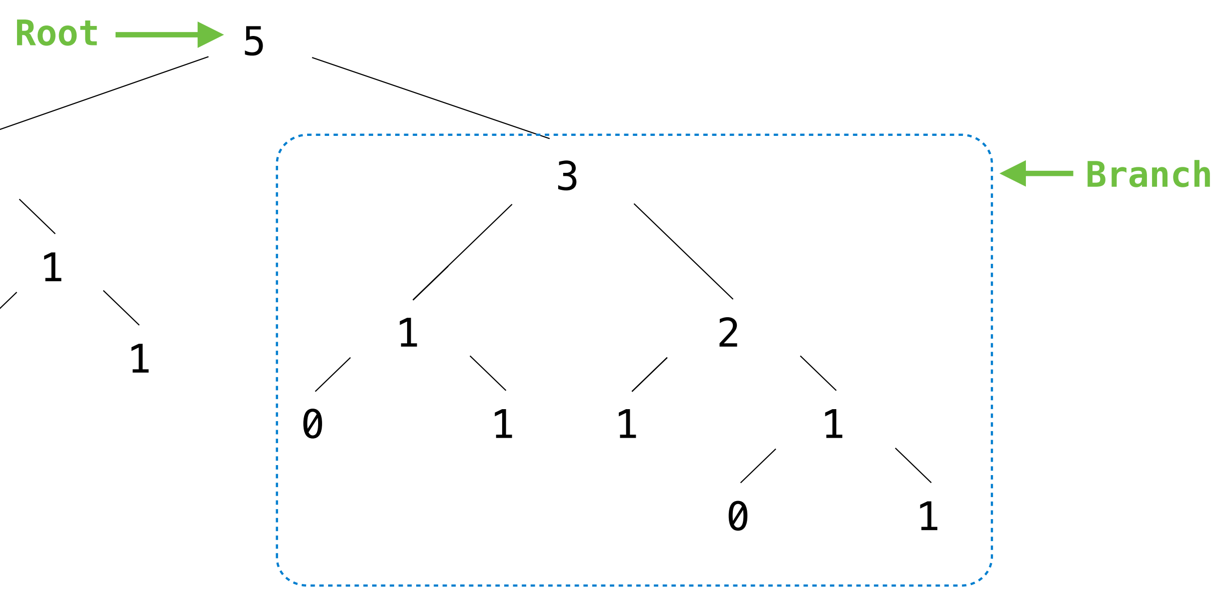

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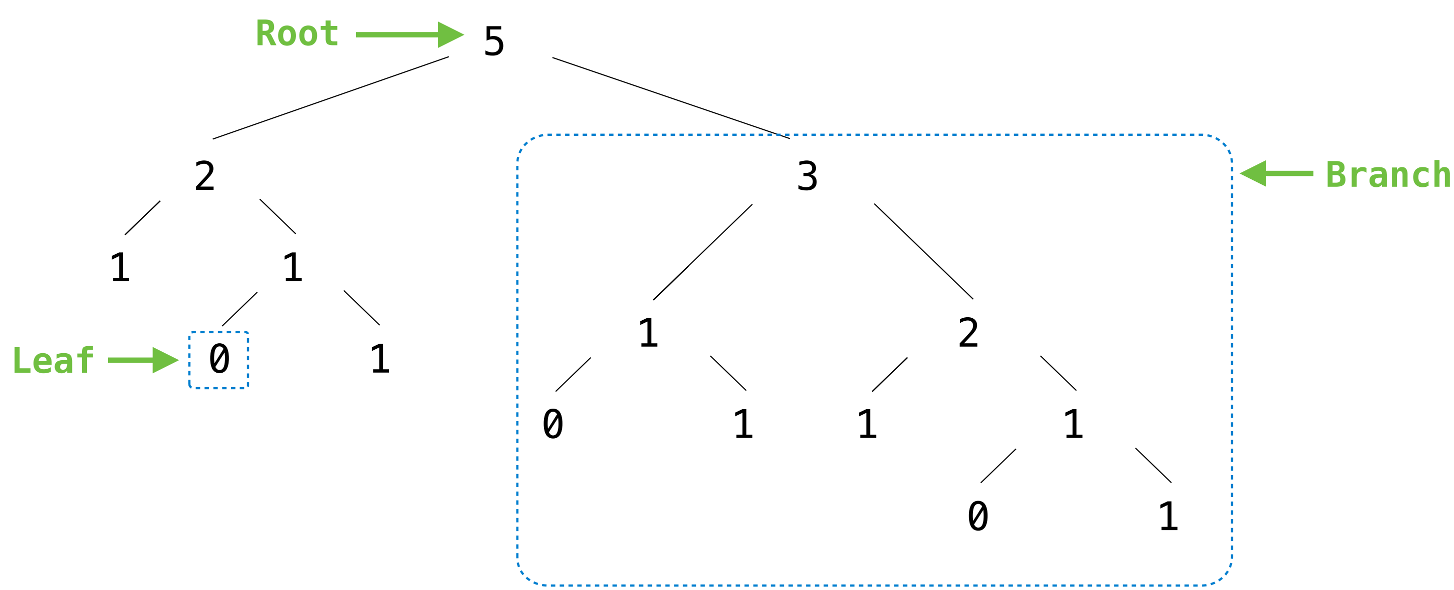

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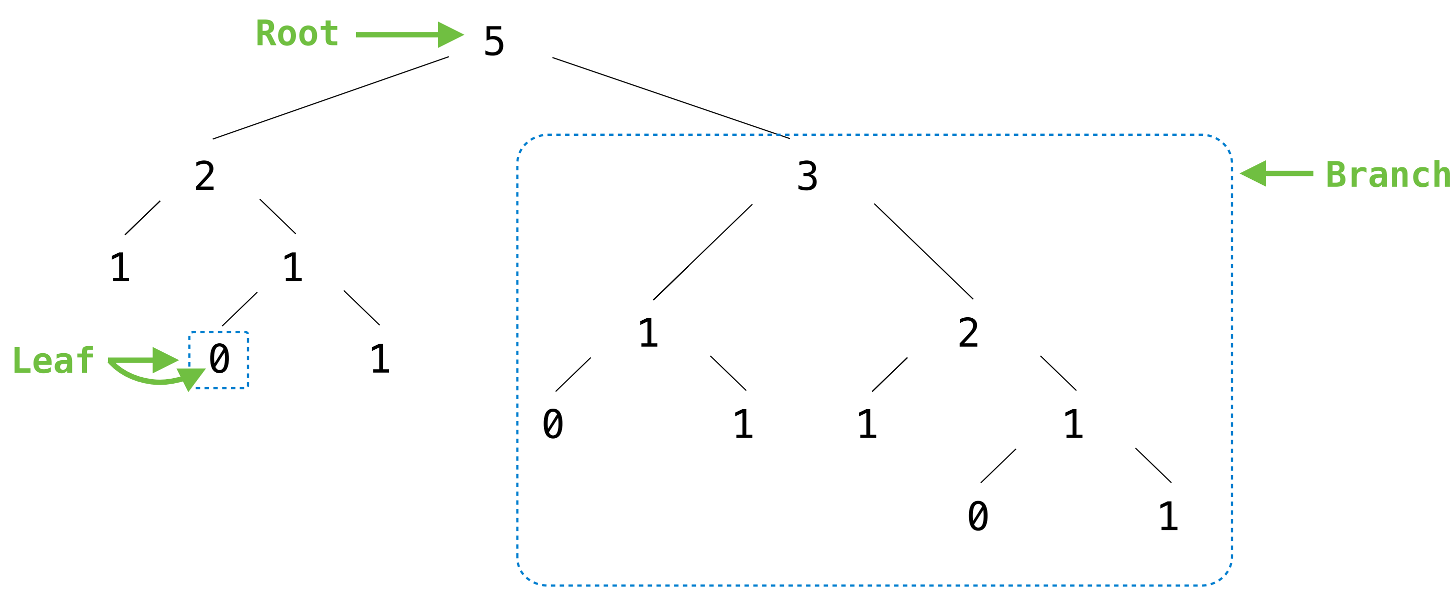

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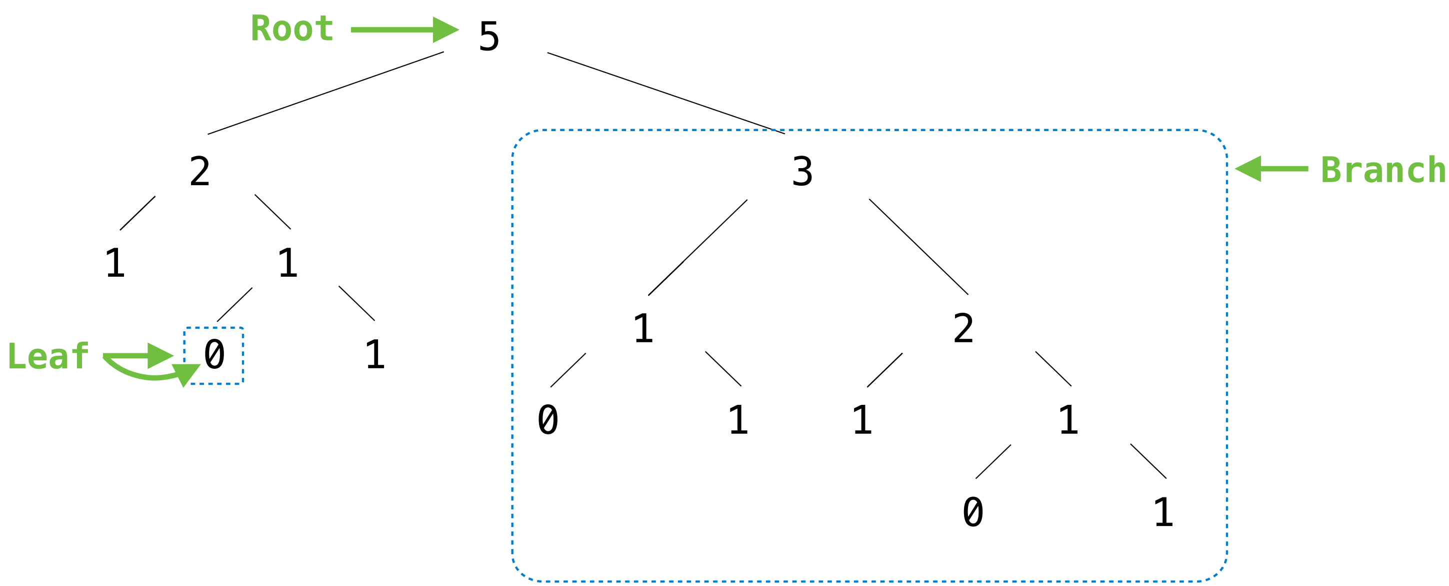

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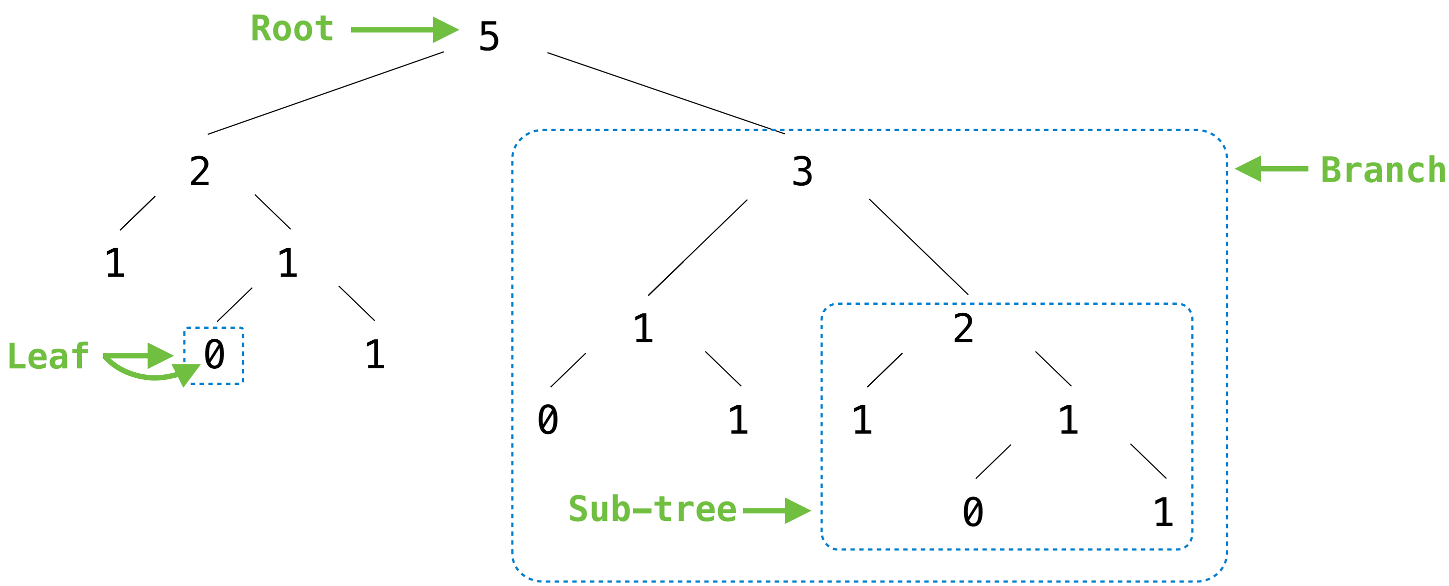

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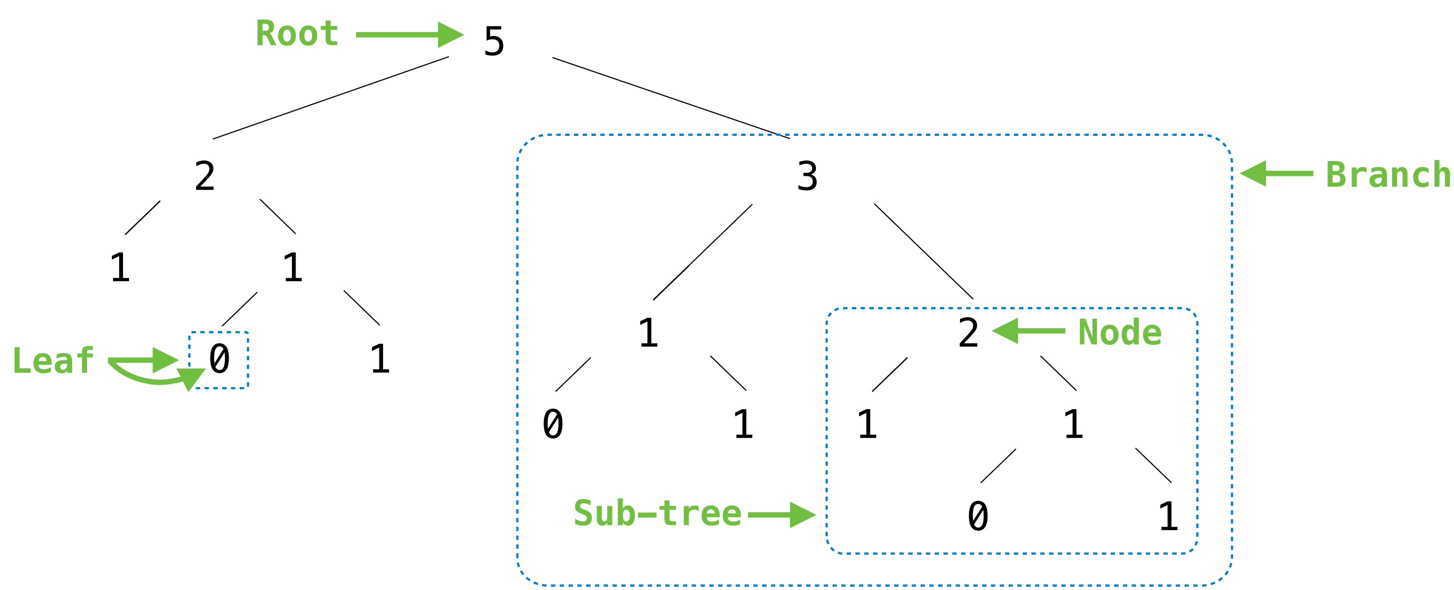

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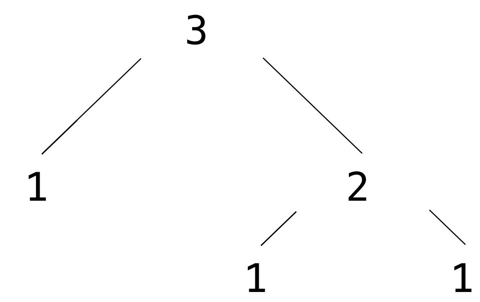

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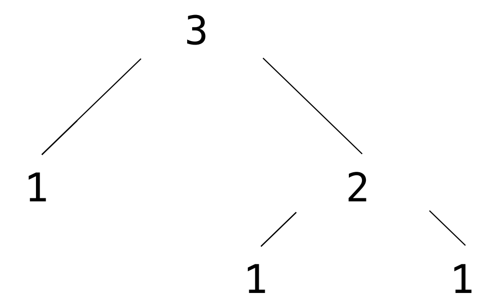


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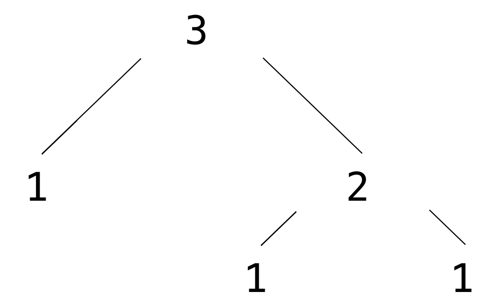


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def tree(root, branches=[]):
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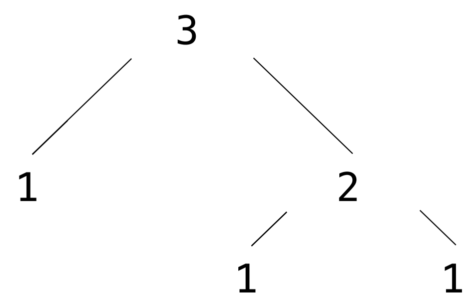


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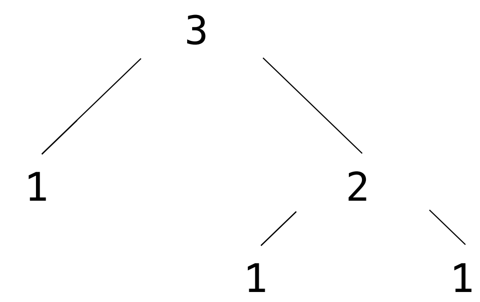

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def tree(root, branches=[]):
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def root(tree):
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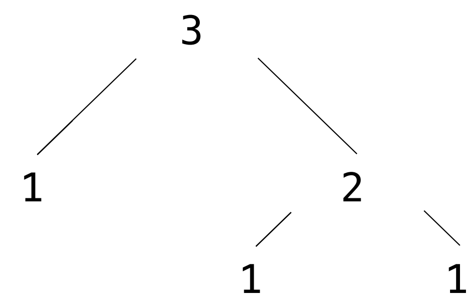


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def tree(root, branches=[]):
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def root(tree):
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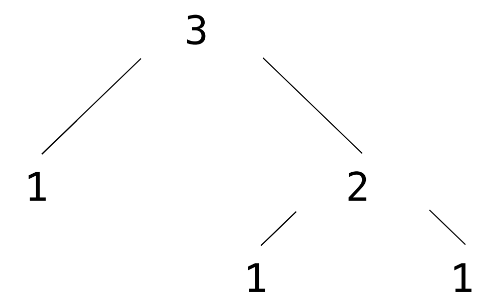


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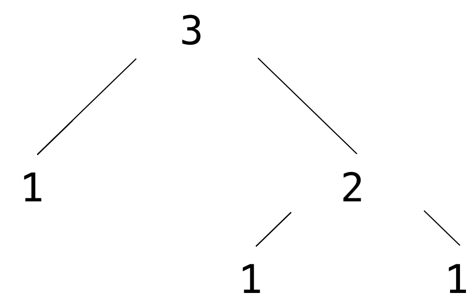


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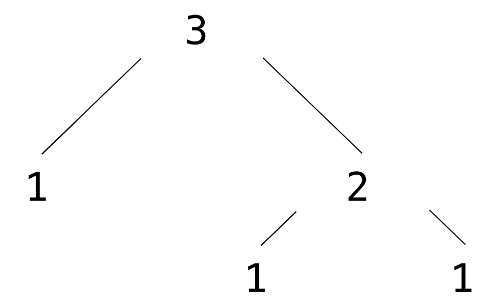


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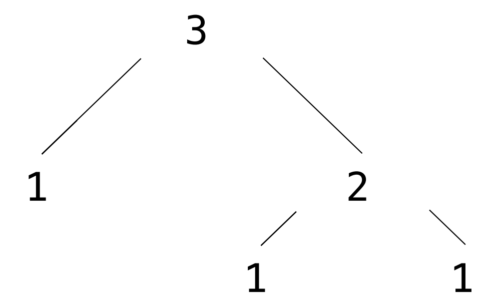


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Creates a list from a sequence of branches

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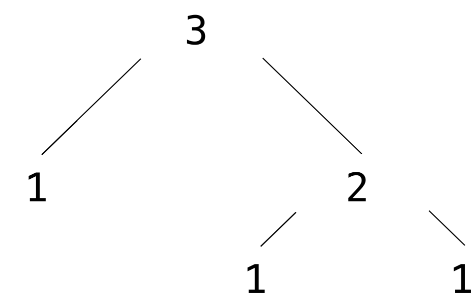


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Verifies the tree definition

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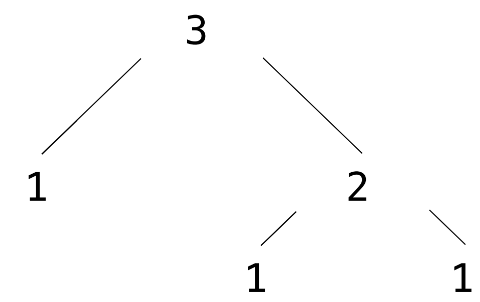


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def is_tree(tree):
    if type(tree) != list or len(tree) < 1:
        return False
    for branch in branches(tree):
        if not is_tree(branch):
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    return True
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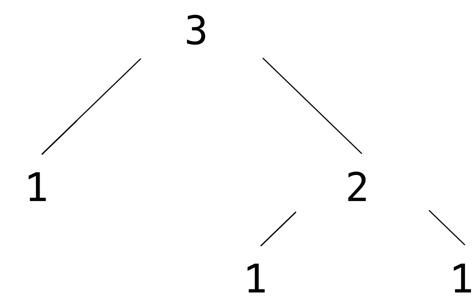


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Verifies that tree is bound to a list

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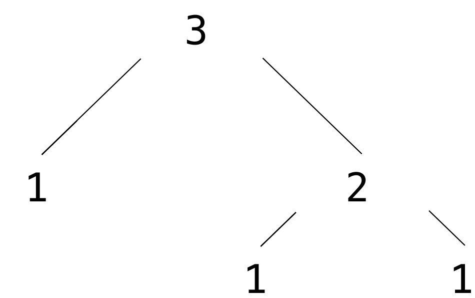


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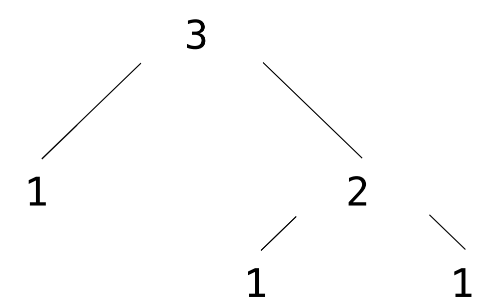


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def is_leaf(tree):
    return not branches(tree)
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Implementing the Tree Abstraction

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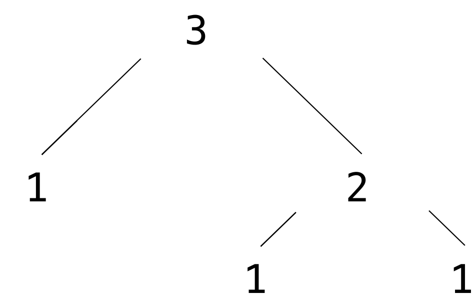


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(Demo)

Tree Processing

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    """Count the leaves of a tree."""
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(Demo)

Discussion Question

Discussion Question

`Implement leaves, which returns a list of the leaf values of a tree`

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Implement leaves, which returns a list of the leaf values of a tree

```
def leaves(tree):
    """Return a list containing the leaves of tree.

    >>> leaves(fib_tree(5))
    [1, 0, 1, 0, 1, 1, 0, 1]
    """
```

Discussion Question

Implement `leaves`, which returns a list of the leaf values of a tree

Hint: If you `sum` a list of lists, you get a list containing the elements of those lists

```
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    """Return a list containing the leaves of tree.

    >>> leaves(fib_tree(5))
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    """
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Discussion Question

Implement `leaves`, which returns a list of the leaf values of a tree

Hint: If you `sum` a list of lists, you get a list containing the elements of those lists

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leaves(tree)                            [s for s in leaves(tree)]

[branches(b) for b in branches(tree)]   [branches(s) for s in leaves(tree)]

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Example: Partition Trees

(Demo)

Interactive Diagram