

# 1 Single Spin Interference

What happens if we take the spin wavefunction of a particle, break it into two pieces and let it interfere with itself? How do you do this? Use a classic 2-slit experiment. You can get strange interference effects.

Imagine the following strange device:

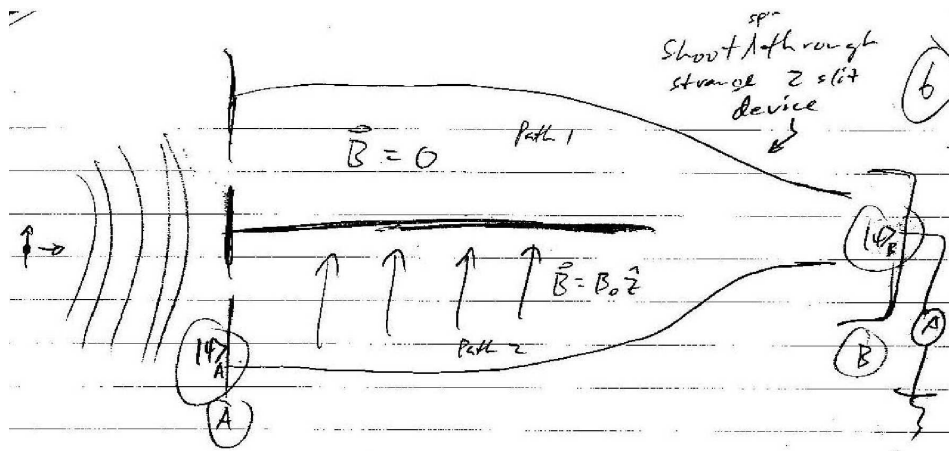


Figure 1:

The particle starts in spin up  $|\psi\rangle_A = |0\rangle$ . So what we will do is shoot spins through the device and measure the number of spins that get through to B:

$$|\psi\rangle_B = e^{-i\hat{H}t/\hbar} = |\psi\rangle_A = |\psi\rangle_{path1} + |\psi\rangle_{path2}$$

This is the classic description of interference where we superpose to quantum states and see if they constructively or destructively interfere. But what are the quantum states for the two paths?

$$|\psi\rangle_{path1} = |0\rangle$$

and

$$|\psi\rangle_{path2} = e^{-i\frac{\hat{S}_z}{\hbar}\Delta\phi}|0\rangle$$

where  $\Delta\phi = \frac{eB_0}{m}\Delta t$  and  $\Delta t$  is the transit time.

Now let's suppose that  $B_o$  and  $\Delta t$  are tuned so that  $\Delta\phi = 2\pi$ . What happens?

Naively, we might think nothing would happen because a  $2\pi$  rotation brings you right back to where you started. However, this would be incorrect because we need to note the following:

$$|\psi\rangle_{path2} = e^{-i\frac{S_z}{\hbar} \cdot 2\pi} |0\rangle = e^{-i\frac{\hbar}{2} \cdot \frac{2\pi}{\hbar}} |0\rangle = e^{-i\pi} |0\rangle$$

So  $|\psi\rangle_{path2} = -|0\rangle$ ! Normally we ignore the “-” sign since it's an overall phase factor, but now we can't since the particle is going to interfere with itself!! So what happens at point B?  $|\psi\rangle_B = |0\rangle + (-|0\rangle) = 0$ .

How many particles get through? ZERO!!!

This has been experimentally observed in what are now classic experiments with neutrons. See H. Rauch et al., Phys. Lett. **64A**, 425 (1975) and S. A. Werner et al., Phys. Rev. Lett. **35** 1053 (1975).

## 2 Spin rotations - state versus expectation value

We have been discussing the effects of a rotation on the state vector, described for a spin 1/2 particle by a vector on the Bloch sphere. Let us summarize. A rotation about, e.g., the  $z$  axis, described by the operator  $R_z(\gamma) = \exp(-\frac{S_z}{\hbar}\gamma)$  rotates the state  $\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$  on the Bloch sphere by  $\gamma$  about the  $z$  axis (recall also problem set 3):

$$R_z(\gamma)[\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle] = e^{-i\gamma/2}[\cos(\frac{\theta}{2})|0\rangle + e^{i\phi+\gamma}\sin(\frac{\theta}{2})|1\rangle]. \quad (1)$$

We see that for  $\gamma = 2\pi$ , the state is returned not to itself, but to the negative of itself, i.e., it has gained a global phase factor of -1. This seems counter-intuitive but is less surprising when we recall that we are talking about a rotation in the state space and not in real space. For the states of a spin 1/2 particle ('spinor states') it is characteristic that a rotation of  $2\pi$  brings a factor of (-1) - a rotation of  $4\pi$  is needed to bring the state back to itself.

Now consider what happens to expectation values of the spin operators, i.e., to  $\langle S_x \rangle$ ,  $\langle S_y \rangle$ ,  $\langle S_z \rangle$ , when the state is rotated. Since these are observables that can be measured in real space (unlike the global phase factor about which is only made observable by an interferometric experiment), we expect that they should behave as vectors in real space under rotations. This is indeed the case and can be verified by direct calculation, as follows.

$$S_x = \frac{\hbar}{2}[|0\rangle\langle 1| + |1\rangle\langle 0|] \quad (2)$$

$$S_y = i\frac{\hbar}{2}[|0\rangle\langle 1| - |1\rangle\langle 0|] \quad (3)$$

$$S_z = \frac{\hbar}{2}[|0\rangle\langle 0| - |1\rangle\langle 1|] \quad (4)$$

Define  $\langle S_x \rangle_\psi = \langle \psi | S_x | \psi \rangle$  and then evaluate the expectation value in the rotated state  $|\psi'\rangle = R_z(\phi)|\psi\rangle$ .

$$\begin{aligned}
\langle S_x \rangle_{\psi'} &= \frac{\hbar}{2} \langle \psi | R_z^\dagger(\phi) S_x R_z(\phi) | \psi \rangle \\
&= \frac{\hbar}{2} \langle \psi | \left[ e^{i\phi/2} |0\rangle \langle 1| e^{i\phi/2} + e^{-i\phi/2} |1\rangle \langle 0| e^{-i\phi/2} \right] | \psi \rangle \\
&= \frac{\hbar}{2} \langle \psi | \frac{(e^{i\phi} + e^{-i\phi})}{2} [|0\rangle \langle 1| + |1\rangle \langle 0|] + i \frac{(e^{i\phi} - e^{-i\phi})}{2} [|0\rangle \langle 1| - |1\rangle \langle 0|] \\
&= \langle S_x \rangle_\psi \cos(\phi) - \langle S_y \rangle_\psi \sin(\phi).
\end{aligned}$$

Similarly, you can show that  $\langle S_y \rangle_{\psi'} = \langle S_x \rangle_\psi \sin(\phi) + \langle S_y \rangle_\psi \cos(\phi)$  and  $\langle S_z \rangle_{\psi'} = \langle S_z \rangle_\psi$ . This transformation corresponds to a real rotation of  $\langle \vec{S} \rangle$  about the  $z$  axis in real space. In this situation, for  $\phi = 2\pi$ , we rotate the expectation value of the spin back onto itself without any sign change.

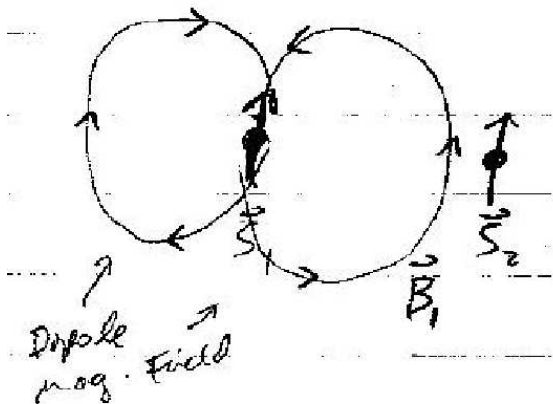
### 3 Entanglement and Spin

So far we've talked about 1 qubit operations (e.g.  $|\psi\rangle \Rightarrow |\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$ ). But what about entanglement? What about when there are more than one qubits?

*Question:* How do we physically create an entangled state of 2 spins?

*Answer:* Must have an interaction between them, i.e. *two* particle Hamiltonian.

How do we create such an interaction and how does it lead to entanglement? Let's start with two physical qubits, say, electrons:



*Claim:* The ground state of this system is an entangled state! Namely,  $|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2)$ , a Bell state!

How do we show this? It's the same old quantum story, solving the Schr. equation.

So what is  $\hat{H}$ ? We must figure out how these electrons interact with each other. What effect could one electron have on the other electron, and vice versa?

Well, we know that an electron has a magnetic dipole moment that is related to its spin. Magnetic dipole moments come up classically when you have current loops, so conceptually we can think of electrons as

little current loops that generate dipolar magnetic fields. But if one electron generates a magnetic field, then the other electron can “feel” that electric field. If you put two electrons close enough together, then we imagine that the two generated fields from electron #1 (#2) will be felt by electron #2 (#1).

What does this look like mathematically, i.e. what is the Hamiltonian? Well, we know  $\vec{\mu} = -\frac{e}{m}\vec{S}$  and  $\hat{H} = \vec{\mu} \cdot \vec{B}$ . But we know that the magnetic field produced by a magnetic moment is proportional to that vector, so we can say  $\vec{B}_1 \propto \vec{S}_1$ , or  $\vec{B}_1 = A\vec{S}_1$  where  $A > 0$ . So,

$$\hat{H} = -\left(-\frac{e}{m}\vec{S}_2\right) \cdot A\vec{S}_1 \quad (5)$$

or equivalently,

$$\hat{H} = C\vec{S}_2 \cdot \vec{S}_1 \quad (6)$$

where  $C > 0$ . This is our electron interaction Hamiltonian. You might wonder what C is. C should be a strong function of the relative position, but we don't want to worry about that now. You can just imagine that C is determined through experiment, but is equal to some value that is yet to be determined. The important part is the  $\vec{S}_1 \cdot \vec{S}_2$  term.

So what is the ground state of this Hamiltonian? Let's use a little trick (whenever you hear “trick” in regards to spins and angular momentum, it's time to bust out the raising/lowering operators).

Consider  $\vec{S}_{Total} = \vec{S}_1 + \vec{S}_2$ , a new operator that should represent the *total* spin of the two electron system. Let's look at some features of this new operator:

$$\vec{S}_T \cdot \vec{S}_T = \hat{S}_T^2 = (\vec{S}_1 + \vec{S}_2) \cdot (\vec{S}_1 + \vec{S}_2) = \hat{S}_1^2 + \hat{S}_2^2 + 2\vec{S}_2 \cdot \vec{S}_1 \quad (7)$$

We see that the dot product  $\vec{S}_2 \cdot \vec{S}_1$  has appeared. Let's solve for this quantity:

$$\vec{S}_2 \cdot \vec{S}_1 = \frac{1}{2}(\hat{S}_T^2 - \hat{S}_1^2 + \hat{S}_2^2) \quad (8)$$

and therefore our interaction Hamiltonian can be expressed as:

$$\hat{H} = \frac{C}{2}(\hat{S}_T^2 - \hat{S}_1^2 + \hat{S}_2^2) \quad (9)$$

So, the ground state will be whatever state minimizes the expectation value of this operator (recall  $\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$ ).

Note that no matter what  $|\psi\rangle$  is,  $\hat{S}_1^2|\psi\rangle = \hbar^2\frac{1}{2}(\frac{1}{2} + 1)|\psi\rangle = \frac{3}{4}\hbar^2|\psi\rangle$ . The same goes for  $\hat{S}_2^2$ , so we can replace both  $\hat{S}_1^2$  and  $\hat{S}_2^2$  with  $\frac{3}{4}\hbar^2$ . We see that the Hamiltonian can be rewritten as:

$$\hat{H} = D\hat{S}_T^2 - F \quad (10)$$

where D and F are constants greater than zero. So what state  $|\psi\rangle$  has the smallest  $\langle\psi|\hat{S}_T^2|\psi\rangle$ ? The best we could hope for is  $|\psi\rangle$  such that  $\langle\psi|\hat{S}_T^2|\psi\rangle = 0$ .

Using spin algebra (see, e.g., lecture 14, from CS191 in Fall 2007) one show that the following state

$$|\psi\rangle_o = \frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2) \quad (11)$$

is an eigenstate of  $\hat{S}_T^2$  with eigenvalue 0. In the next section we provide general rules for obtaining the possible spin states from combining systems that each possess angular momentum.

We conclude from this that we can experimentally create a Bell state by putting 2 spins next to each other and then providing a perturbation such that they fall into the ground state. This is just a first suggestion for how to make an entangled state, given an interaction between 2 spins.

## 4 Two spins: addition of angular momenta

Consider two angular momenta  $\vec{L}_1$  and  $\vec{L}_2$ . Our treatment is general, and does not distinguish orbital from spin angular momentum. We can make a composite state  $|L_1, L_2, m_1, m_2\rangle$  since the four operators  $L_1^2, L_2^2, L_{1z}$  and  $L_{2z}$  are mutually commuting. We refer to this as the 'uncoupled representation'. But we can also measure the total angular momentum and its z-projection,  $\vec{L}^2 = (\vec{L}_1 + \vec{L}_2)^2$  and  $L_z = L_{z1} + L_{z2}$ . Furthermore, these two operators commute with  $L_1$  and  $L_2$  (check). So we can also form the state  $|L, m, L_1, L_2\rangle$  where  $m = L_z$ . We refer to this as the 'coupled representation'.

Now the question is, what are the allowed values of  $L$  and  $m$ ? Well, the maximum value of  $m$  must be  $m_{max} = m_{1max} + m_{2max} = L_1 + L_2$ . Hence the maximum allowed value of  $L$  is also equal to  $L_1 + L_2$ . This  $L$  value will have  $2L + 1$  possible  $m$  values associated with it.

What other states of  $L$  are possible? We can use a state counting argument to find them, together with the requirement that  $L$  change by integral values only. In the uncoupled representation we have a total of  $(2L_1 + 1)(2L_2 + 1)$  states. On changing to the coupled representation we are just relabeling states and must preserve the dimensionality of the space. So we must have

$$\sum_{L_{min}}^{L_1+L_2} (2L + 1) = (2L_1 + 1)(2L_2 + 1).$$

This is satisfied if  $L_{min} = |L_1 - L_2|$ . Hence our allowed values of total angular momentum are given by

$$\begin{aligned} L &= |L_1 - L_2|, |L_1 - L_2| + 1, \dots, L_1 + L_2 \\ m &= -L, -L + 1, \dots, +L. \end{aligned}$$

Now lets evaluate this for 2 spins, e.g., 2 electrons.  $S_1 = S_2 = 1/2$ . Hence the allowed values of total spin are  $S = 1, S = 0$ . The  $S = 1$  state has three values of  $m = -1, 0, +1$  associated with it and is called a spin triplet. The  $S = 0$  state has one value of  $m = 0$  associated with it and is called a spin singlet.