1 Readings

Benenti, Casati, and Strini:

No Cloning Ch.4.2

Teleportation Ch. 4.5

2 No Cloning Theorem

A quantum operation which copied states would be very useful. For example, we considered the following problem in Homework 1: Given an unknown quantum state, either $|\phi\rangle$ or $|\psi\rangle$, use a measurement to guess which one. If $|\phi\rangle$ and $|\psi\rangle$ are not orthogonal, then no measurement perfectly distinguishes them, and we always have some constant probability of error. However, if we could make many copies of the unknown state, then we could repeat the optimal measurement many times, and make the probability of error arbitrarily small. The no cloning theorem says that this isn't physically possible. Only sets of mutually orthogonal states can be copied by a single unitary operator.

There are two ways to prove the no cloning theorem. The first follows from the norm preserving property of the inner product, the second from the linearity of quantum mechanics.

No Cloning Assume we have a unitary operator U_{cl} and two quantum states $|\phi\rangle$ and $|\psi\rangle$ which U_{cl} copies, i.e.,

$$\begin{aligned} |\phi\rangle \otimes |0\rangle &\xrightarrow{U_{cl}} |\phi\rangle \otimes |\phi\rangle \\ |\psi\rangle \otimes |0\rangle &\xrightarrow{U_{cl}} |\psi\rangle \otimes |\psi\rangle \ . \end{aligned}$$

Then $\langle\phi|\psi\rangle$ is 0 or 1.

Proof 1: $\langle\phi|\psi\rangle = (\langle\phi| \otimes \langle 0|)(|\psi\rangle \otimes |0\rangle) = (\langle\phi| \otimes \langle\phi|)(|\psi\rangle \otimes |\psi\rangle) = \langle\phi|\psi\rangle^2$. In the second equality we used the fact that U, being unitary, preserves inner products. □

Proof 2: Suppose there exists a unitary operator U_{cl} that can indeed clone an unknown quantum state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then

$$\begin{aligned} |\phi\rangle|0\rangle \xrightarrow{Ucl} |\phi\rangle|\phi\rangle &= (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha^2|00\rangle + \beta\alpha|10\rangle + \alpha\beta|01\rangle + \beta^2|11\rangle \end{aligned}$$

But now if we use U_{cl} to clone the expansion of $|\phi\rangle$, we arrive at a different state:

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \xrightarrow{U_{cl}} \alpha|00\rangle + \beta|11\rangle \, .$$

Here there are no cross terms. Thus we have a contradiction and therefore there cannot exist such a unitary operator U_{cl}. □

Note that it is however possible to clone a known state such as $|0\rangle$ and $|1\rangle$.

3 Teleportation

Contrary to its sci-fi counterpart, quantum teleportation is rather mundane. Quantum teleportation is a means to replace the *state* of one qubit with that of another. It gets its out-of-this-world name from the fact that the state is "transmitted" by setting up an entangled state-space of three qubits and then removing two qubits from the entanglement (via measurement). Since the information of the source qubit is preserved by these measurements that "information" (i.e. state) ends up in the final third, destination qubit. This occurs, however, without the source (first) and destination (third) qubit ever directly interacting. The interaction occurs via entanglement. Figure 1 (see below) shows the set up for quantum teleportation, and Figure 2 (see below) presents a quantum circuit implementing teleportation of a one-qubit state.

Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$ and given an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the state of the entire system is:

$$\frac{1}{\sqrt{2}}\left[a|0\rangle\left(|00\rangle+|11\rangle\right)+b|1\rangle\left(|00\rangle+|11\rangle\right)\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} a \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ b \end{bmatrix}$$

Perform the *CNOT* operation and you obtain

$$\frac{1}{\sqrt{2}}\left[a|0\rangle\left(|00\rangle+|11\rangle\right)+b|1\rangle\left(|10\rangle+|01\rangle\right)\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} a \\ 0 \\ 0 \\ a \\ 0 \\ b \\ b \\ 0 \end{bmatrix}$$

Next we apply the H gate. However, as an aside, lets examine what happens when we apply the H gate to $|0\rangle$ and to $|1\rangle$. Recall that:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, applying H to our system we have:

$$\left|\varphi\right\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}a\left(\left|0\right\rangle+\left|1\right\rangle\right)\left(\left|00\right\rangle+\left|11\right\rangle\right)+\frac{1}{\sqrt{2}}b\left(\left|0\right\rangle-\left|1\right\rangle\right)\left(\left|10\right\rangle+\left|01\right\rangle\right)\right] = \frac{1}{2}\begin{bmatrix} a \\ b \\ b \\ a \\ a \\ -b \\ -b \\ a \end{bmatrix}$$

We can rewrite this expression as:

$$\begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} b \\ a \end{bmatrix} \\ \begin{bmatrix} a \\ -b \end{bmatrix} \\ \begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = \frac{1}{2}\left[\left|00\right\rangle\left(a\left|0\right\rangle+b\left|1\right\rangle\right)+\left|01\right\rangle\left(a\left|1\right\rangle+b\left|0\right\rangle\right)+\left|10\right\rangle\left(a\left|0\right\rangle-b\left|1\right\rangle\right)+\left|11\right\rangle\left(a\left|1\right\rangle-b\left|0\right\rangle\right)\right],$$

which we can shorten to:

$$\frac{1}{2}\left[\left|00\right\rangle\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left|\psi\right\rangle+\left|01\right\rangle\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left|\psi\right\rangle+\left|10\right\rangle\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left|\psi\right\rangle+\left|11\right\rangle i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\left|\psi\right\rangle\right].$$

We recognise that the third qubit is now in a state given by the action of one of the well-known Pauli operators I,X,Y,Z on the unknown initial state $\left|\psi\right\rangle$ of qubit 1. The state of qubit 3 can also be written as:

$$\frac{1}{2}\left[\left|00\right\rangle I\left|\psi\right\rangle+\left|01\right\rangle X\left|\psi\right\rangle+\left|10\right\rangle Z\left|\psi\right\rangle+\left|11\right\rangle iY\left|\psi\right\rangle\right]$$

and alternatively as:

$$\left|\varphi\right\rangle = \frac{1}{2}\left[\left|00\right\rangle I\left|\psi\right\rangle+\left|01\right\rangle X\left|\psi\right\rangle+\left|10\right\rangle Z\left|\psi\right\rangle+\left|11\right\rangle XZ\left|\psi\right\rangle\right].$$

Notice that the two-qubit state of qubits 1 and 2 is different in each term. This result implies that we can measure the first and second qubit and obtain two classical bits which will tell us what transform was applied to the third qubit. Thus we can subsequently "fixup" the third qubit once we know the classical outcome of the measurement of the first two qubits. This fixup is fairly straightforward, either applying nothing, X, Z or both X and Z. (Recall that $X^2 = Y^2 = Z^2 = I$.)

Lets work through an example. Suppose the result of measuring qubits 1 and 2 is 10. Then from the above, qubit 3 must be in the state $Z\left|\psi\right\rangle$. The matrix representing the measurement operator is

$$M_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$P(10) = \left\langle\varphi\right|M_{10}^{\dagger}M_{10}\left|\varphi\right\rangle = \left\langle\varphi\right|M_{10}\left|\varphi\right\rangle$, since here $M_{10}^{\dagger}M_{10}$. Thus:

$$M_{10}\left|\varphi\right\rangle = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ -b \\ 0 \\ 0 \end{bmatrix}$$

Hence: $\left\langle\varphi\right|M_{10}\left|\varphi\right\rangle = \frac{1}{2}\left[a,b,b,a,a,-b,-b,a\right]\frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ -b \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4}\left[a\cdot a^* + b\cdot b^*\right]$

Recall that by definition of a qubit we know that $a\cdot a^* + b\cdot b^* = 1$, hence the probability of measuring 01 is $1/4$. The same is true for the other outcomes.

What have we done? We have inserted an unknown single qubit quantum state into a system of 3 qubits where the other two qubits shared some entanglement. We carried out some unitary operations on qubits 1 and 2, and then measured out these two qubits. The result is that the unknown quantum state has been migrated through entanglement to qubit 3, where it is can be recovered by making a single qubit unitary operation dependent on the two measured values from qubits 1 and 2.

Quantum teleportation has been termed "disembodied transfer of quantum information from one place to another" (S. Braunstein). It does not violate relativity: the source sends only classical information (the result of the measurements of qubits 1 and 2) and this must be done by conventional means, e.g., optical fiber. The source sends no information about the quantum state. Neither does it violate the no-cloning theorem since the quantum state is destroyed at the source and created at the destination. ie.,

$$\left|\psi\right\rangle\left|0\right\rangle \longrightarrow \left|x\right\rangle\left|\psi\right\rangle.$$

Here $\left|x\right\rangle$ is the state of qubit 1 after measurement.

Teleportation illustrates an equivalence between quantum bits (qubits), entanglement bits (e-bits), and classical bits (c-bits):

1 qubit $\equiv$ 1 e-bit + 2 c-bits

Note the difference between making a FAX copy and creating a copy by quantum teleportation. With a FAX, i) the original is preserved, and ii) only a partial copy is obtained. With quantum teleportation, i) the original state is destroyed (but not the qubit), and ii) an exact copy of the quantum state results.

Accessible sources on quantum teleportation:

IBM web page: $\mathrm{http://wwww.research.ibm.com/quantum_info/teleportation}$

C. Caves, Science 282 (23 October) 1998, p. 637

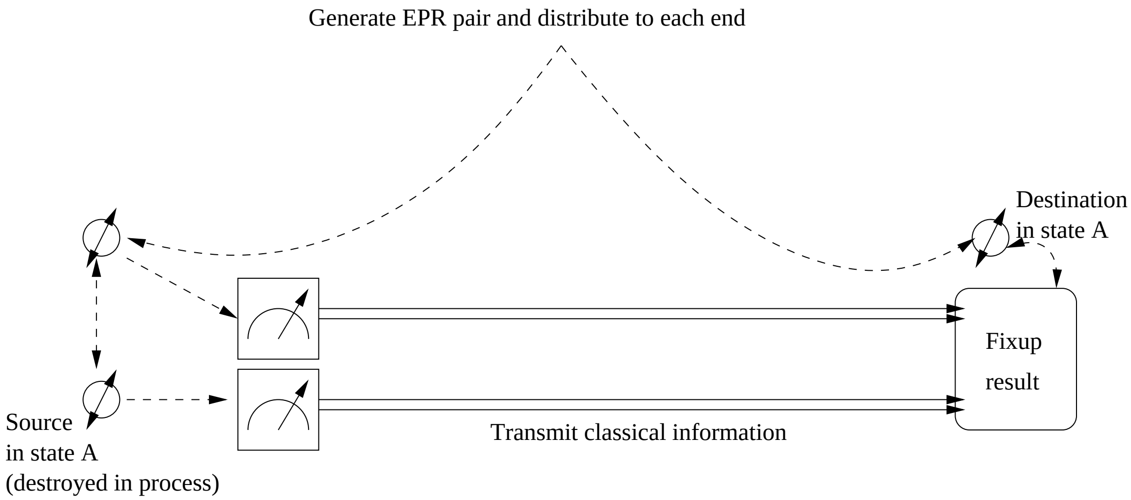

Figure 1: Teleportation requires pre-transmitting an EPR pair to the source and destination. The qubit containing the state to be "teleported" then interacts with one half of this EPR pair, creating a joint state space. Unitaries are performed in this joint state space and then these 2 qubits are measured. The resulting classical information of the measurement outcome is transmitted to the destination. This classical information is used to "fixup" the destination qubit with single qubit unitaries.

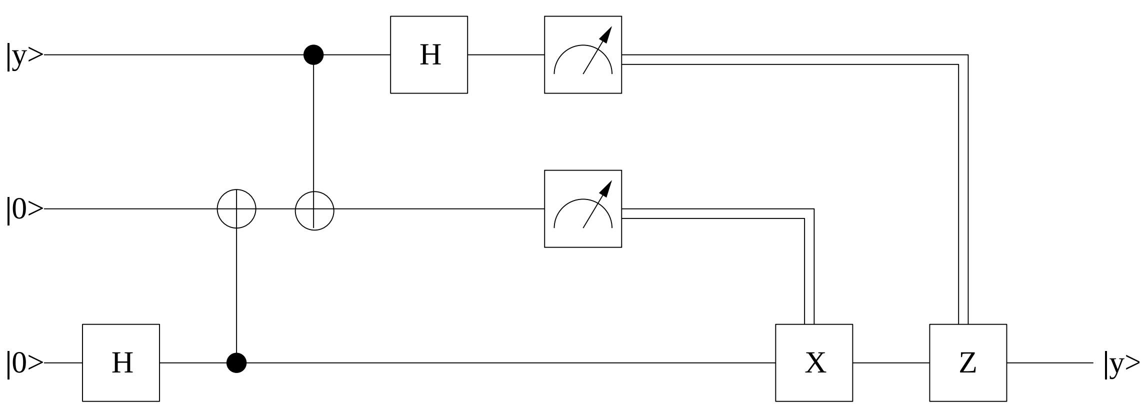

Figure 2: Quantum circuit implementing teleportation. The first two operations on qubits 2 and 3 at the bottom right form the EPR pair. Note that in this diagram single lines represent quantum data while double lines represent classical information.