CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin HW 5

Due ~~Monday July 27~~ Tuesday July 28 at Noon

1. **Best-of-Five Series — Go Bears!** (60 points, 3/3/10/5/10/5/3/3/3/3/3/3/3/3 points for each part)

   A best-of-five series is the competition between two teams where a team wins the series by winning three games in the series. After a team wins three games, the following games will not be played. An outcome of a best-of-five series is defined by the result of each game, *e.g.*, WWW, WWLW, WLWW, and WLWLW are all different outcomes for a team. Now, two basketball teams, California Golden Bears and Stanford Cardinal, are going to play a best-of-five series.

   (a) How many possible outcomes for Golden Bears to win the series with 3 wins and 2 losses?

   (b) How many possible outcomes for Golden Bears to win the series?

   (c) Prove $\sum_{i=0}^{n} \binom{n+i}{i} = \binom{2n+1}{n+1}$ with explicit calculations ($n \geq 0$).

   (d) If it is a best-of-$(2n+1)$ series, use the above equality to prove that there are $\binom{2n+1}{n+1}$ outcomes for Golden Bears to win the series.

   (e) If it is a best-of-$(2n+1)$ series, explain "there are $\binom{2n+1}{n+1}$ outcomes for Golden Bears to win the series" without any explicit calculation.

   (f) If it is a best-of-(n_1+n_2-1) series where Golden Bears win the series by winning n_1 games in the series, and Cardinal wins the series by winning n_2 games in the series, explain "there are $\binom{n_1+n_2-1}{n_1}$ outcomes for Golden Bears to win the series" without any explicit calculation.

   It is believed that Golden Bears have a 0.7 probability to win each game, if the game is played.

   (g) What is the probability for Golden Bears to win the series with 3 wins and 2 losses?

   (h) What is the probability for Golden Bears to win the series?

   (i) The captain of Cardinal proposes to change the series from best-of-five to best-of-three. Is this change advantageous to Golden Bears or Cardinal? Prove your answer.

   Now, let's consider some conditional probabilities. We still assume that Golden Bears have a 0.7 probability to win each game, if the game is played.

   (j) $\Pr[$ (Golden Bears win the series) $\mid$ (Golden Bears win the first game) $] = ?$

   (k) $\Pr[$ (Golden Bears win the first game) $\mid$ (Golden Bears win the series) $] = ?$

   (l) $\Pr[$ Golden Bears win the fifth game $] = ?$ (We are considering this before the series starts.)

   (m) $\Pr[$ (Golden Bears win the series) $\mid$ (Golden Bears win the fifth game) $] = ?$

   (n) $\Pr[$ (Golden Bears win the fifth game) $\mid$ (Golden Bears win the series) $] = ?$

2. **Bayesian Inference and Pancakes** (15 points, 3 points for each part)

   Kunal is making golden-brown pancakes and you are hungry!

   (a) Kunal serves up a stack of 3 pancakes, but he forgot to butter the pan! Pancake A is perfect (golden-brown on both sides), pancake B is burnt on one side, and pancake C is burnt on both sides. The top of the stack is burnt. What's the probability that the other side of the top pancake is also burnt? Justify your answer.

   (b) Kunal agrees that a burnt pancake on top of the stack looks un-appetizing, and suggests flipping the stack over. In the same situation as before, what's the probability that the pancake side touching the plate is burnt?

   (c) Suppose Kunal makes a stack of n pancakes such that x pancakes are burnt on both sides and y pancakes are burnt on one side. If the top of the stack is burnt, what's the probability that the other side of the top pancake is also burnt? What if the top of the stack is golden-brown? Justify your answer.

   (d) You asked for chocolate chips, so Kunal adds lots of chocolate chips to the batter. He makes a stack of m pancakes next to the stack of n pancakes from before. However, the k-th pancake ($1 \leq k \leq m$) in the new stack only has a k/m chance of having chocolate chips (independent from the rest of the pancakes). If you choose a pancake randomly from either stack, what's the probability that you get chocolate chips?

   (e) Kunal realizes that the top few pancakes in the new stack don't really have chocolate chips in them. He shifts the top 10 pancakes from that stack (those with the smallest chance of chocolate chips) to the old stack. Given you randomly choose a pancake and it has chocolate chips, what's the probability it came from the new stack?

3. **Error Correction Codes with Probability** (17 points, 3/3/3/3/5 points for each part)

   Alice wishes to send $n = 5$ packets to Bob. However, due to channel noise, there is a 10% chance ($p = 0.1$) for each packet to be corrupted (general error) during transmission. Therefore, Alice wants to transmit m additional packets so Bob can correct potential errors. Bob uses the Berlekamp-Welch Algorithm in the lecture note to fix the errors.

   (a) If $m = 2$, what is the probability that Bob gets Alice's message successfully?

   (b) If $m = 3$, what is the probability that Bob gets Alice's message successfully?

   (c) By the above results, is adding more packets always helpful in this case?

   (d) To guarantee that the probability that Bob gets Alice's message successfully is at least 0.9, how many additional packets does Alice need to transmit?

   Now, we are considering a general case where $n \geq 0$, $m \geq 0$, and $0 \leq p \leq 1$, where p is the probability for each packet to be corrupted.

   (e) Prove it or provide a counterexample: For any n, m, p, transmitting $m + 2$ additional packets is always at least as good as (never worse than) transmitting m additional packets, in terms of the probability that Bob gets Alice's message successfully.

4. **Phase Two** (12 points, 3 points for each part)

   Your Tele-BEARS appointment is tomorrow, but unfortunately one of the classes you need and all its sections and labs are full. The first class has four sections. To enroll in the first class, you must enroll in exactly one section.

   (a) For the first class, Tele-BEARS (miraculously) lets you waitlist all four sections simultaneously (if you please). Assume the probability that you will be able to get off the waitlist for that given section is $1/(n+2)$, where n is the number of people ahead of you on the waitlist. During your appointment, you notice Section 1 has a waitlist of 3 people, Section 2's waitlist has 7 people, Section 3's has 8 people, and Section 4's has 5 people. What is the expected number of sections you will get into if you waitlist all four?

   (b) Suppose Section 1 is at 8am, so you decide not to waitlist it. What is the new expected number of sections you will get into if you waitlist all but Section 1?

   (c) Following Part (a), the professor surprises everyone by deciding to add labs to the course which immediately fill up before your Tele-BEARS appointment. He adds two three hour labs — Lab 1 and Lab 2. Lab 1 has a time conflict with Section 1, and Lab 2 has a time conflict with Section 3. The probability of getting into a lab with n people ahead of you on the waistlist is $1/(n+1)$ and is independent of the probability of getting into any section. There are 10 people on the waitlist for Lab 1 and 12 people on the waitlist for Lab 2 at the time of your Tele-BEARS appointment. Unfortunately, you can only waitlist one of the labs at a time, and to enroll or waitlist any section you MUST be enrolled in or on the waitlist of a lab first, and there must not be any time conflicts. What lab should you waitlist to maximize the expected number of sections you will get into? You MUST get in!!! Justify your answer.

   (d) For your second class, suppose there are k different sections and students are only allowed to waitlist one of them. All k sections are full, and there are m students who want to waitlist the course. If students waitlist for a random section, what is the expected number of sections which will have students on their waitlists?

5. **Random Variable and Expectation** (16 points, 3/3/10 points for each part)

   (a) Given $\Pr[X=1]=\frac{1}{5}, \Pr[X=2]=\frac{4}{5}$, what is $\mathbb{E}(X)$?

   (b) Given $\Pr[X=1]=\frac{1}{14}, \Pr[X=2]=\frac{4}{14}, \Pr[X=3]=\frac{9}{14}$, what is $\mathbb{E}(X)$?

   (c) Assume n is a positive integer. A random variable X has n possible values $\{1,2,\ldots,n\}$, and

   $$\frac{\Pr[X=i]}{\Pr[X=1]}=i^2$$

   for any $i \in \{1,2,\ldots,n\}$. Prove that $\mathbb{E}(X)=\frac{3n(n+1)}{2(2n+1)}$.