Settling Time

Obtain expressions for constant (V_{ss}(s)) and settling time (T_s) as functions of phase margin, \( \phi_m \):

\[
\begin{align*}
A(s) &= \frac{N_o(s)}{N_i(s)} = \frac{a(s)}{1+a(s)} = \frac{a(s)}{s^2 + (\omega_1 + \omega_2)s + \omega_1 \omega_2} = \frac{A_o \omega_1^2}{s^2 + 2s \omega_m + \omega_m^2} \\
&= \frac{A_o \omega_1^2}{s^2 + (\omega_m + \omega_2)s + \omega_1 \omega_2} \\
\frac{a(s)}{s} &= \frac{a(s)}{(s^2 + \omega_m^2)(s^2 + \omega_2^2)} \\
\text{where by direct comparison:} \\
\omega_m &= \sqrt{\omega_1 \omega_2 (1+\alpha \beta)} \\
2\omega_m &= \omega_1 + \omega_2 \\
A_o \omega_1^2 &= A \omega_2^2 \\
A_o &= \frac{a_0}{1+\alpha \beta}
\end{align*}
\]

**Properties of the General Laplace Biquad Transfer Function (a very well studied function!):**

- **Root locus for Bode Chart:** poles move as the loop gain increases;
  \( \Rightarrow \) they become complex
  poles for \( \beta \) small \( \Rightarrow \) \( \omega \approx \frac{\pi}{\beta} \)

- **Time domain behavior:**
  \[
  N_o(t) = A_o V_i \left( 1 - \frac{1}{\sqrt{1-\frac{\pi^2}{\beta^2}}} e^{-\pi t} \sin\left(\sqrt{1-\frac{\pi^2}{\beta^2}} t + \varphi \right) \right), \text{ where } \varphi = \tan^{-1}\left[\frac{\pi}{\beta} \right]
  \]
  \[
  \text{For } s < 1 \text{ (for } \beta \ll \omega_o) \Rightarrow \text{ % Overshoot } = \frac{\text{Peak Value} - \text{Final Value}}{\text{Final Value}} = \exp\left(-\frac{\pi^2}{\beta^2}\right) \Rightarrow V_{\text{overshoot}} = V_0 \exp\left(-\frac{\pi^2}{\beta^2}\right)
  \]
EE 140

\( T_s = f(t_m) \)
s = \text{out freq. at which } |\alpha(j\omega)| = 1 \rightarrow \omega_s.

\alpha(j\omega) = \frac{\omega}{\omega^2 + j2\zeta\omega_0} \Rightarrow |\alpha(j\omega)| = \frac{\omega_s}{\sqrt{\omega_s^2 + 4\zeta^2\omega_0^2}}

\left[|\alpha(j\omega)| = 1 \Rightarrow \frac{\omega_s^2}{\omega_s^2 + 4\zeta^2\omega_0^2} = 1 \rightarrow \omega_s^2 + 4\zeta^2\omega_0^2 - \omega_s = 0\right]

\text{Solve quadratic: } \omega_s = \omega_0\left[\sqrt{5\zeta^2 + 1} - \zeta^2\right]

\text{Find expression for phase:}

\psi = -\tan^{-1}\left(\frac{\omega_0}{2\zeta}\right) - \tan^{-1}\left(\frac{\omega}{2\zeta\omega_0}\right)

\text{For phase margin:}

\phi_m = 180^\circ + \phi = 90^\circ - \tan^{-1}\left(\frac{\omega_0}{2\zeta\omega_0}\right)

\tan\left(90^\circ - \phi_m\right) = \frac{\omega_s}{2\zeta\omega_0}

\left[\tan\left(\phi_m\right) = \frac{1}{\tan\left(90^\circ - \phi_m\right)} = \frac{\omega}{2\zeta\omega_0}\right]

\text{Thus:}

\psi = \frac{1}{2}\zeta \angle \omega_0 \tan \phi_m

\Rightarrow \psi = \frac{z}{\omega_0 \tan \phi_m}\left[\text{Vernadsky}\right]

(ignoring effects of slewing)

\Rightarrow \text{But here's more to it than this. See the following papers:}
