Homework 4 - Solutions

Note: Each part of each problem is worth 3 points and the homework is worth a total of 42 points.

1. State Space Representation To Transfer Function
   Find the transfer function and poles of the system represented in state space below.

   \[
   \dot{x} = \begin{bmatrix} 8 & -4 & 1 \\ -3 & 2 & 0 \\ 5 & 7 & -9 \end{bmatrix} x + \begin{bmatrix} -4 \\ -3 \\ 4 \end{bmatrix} u(t)
   \]

   \[
y = \begin{bmatrix} 2 & 8 & -3 \end{bmatrix} x; \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
   \]

   Solution: \[G(s) = C(sI - A)^{-1}B\]

   \[
   (sI - A)^{-1} = \frac{1}{s^3 - s^2 - 91s + 67} \begin{bmatrix}
   (s - 2)(s + 9) & -(4s + 29) & (s - 2) \\
   -3s - 27 & (s^2 + s - 77) & -3 \\
   5s - 31 & 7s - 76 & (s^2 - 10s + 4)
   \end{bmatrix}
   \]

   \[
   C(sI - A)^{-1}B = \frac{1}{s^3 - s^2 - 91s + 67} \begin{bmatrix}
   2 & 8 & -3 \\
   5s - 31 & 7s - 76 & (s^2 - 10s + 4)
   \end{bmatrix}
   \]

   \[
   G(s) = \frac{-44s^2 + 291s + 1814}{s^3 - s^2 - 91s + 67}
   \]

2. Transfer Function Analysis For Mechanical Systems
   For the system shown below, do the following:
   (a) Find the transfer function \(G(s) = X(s)/F(s)\).
   (b) Find \(\zeta, \omega_n, \%\ OS, T_s, T_p\ and\ T_r\).

   ![Mechanical System Diagram]

   Solution:
   (a) Writing the equation of motion yields, \((5s^2 + 5s + 28)X(s) = F(s)\). Solving the transfer function,

   \[
   \frac{X(s)}{F(s)} = \frac{1}{5s^2 + 5s + 28} = \frac{1}{s^2 + s + \frac{28}{5}}
   \]
(b) Clearly, $\omega_n^2 = 28/5$ rad/s and $2\zeta\omega_n = 1$. Therefore, $\omega_n = 2.37, \zeta = 0.211.$

\[
T_s = \frac{4}{\zeta\omega_n} = 8.01 \text{ sec}
\]
\[
T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 1.36 \text{ sec}
\]
\[
\%OS = e^{-\zeta\pi / \sqrt{1 - \zeta^2}} \times 100 = 50.7\%
\]
\[
T_r = \frac{1}{\omega_n} (1.76\zeta^3 - 0.417\zeta^2 + 1.093\zeta + 1) = 0.514 \text{ sec}
\]

3. **Transfer Function From Unit Step Response**

For each of the unit step responses shown below, find the transfer function of the system.

![Response Graphs](attachment:response_graphs.png)

**Solution:**

(a) This is a first-order system of the form: $G(s) = \frac{K}{s + a}$. Using the graph, we can estimate the time constant as $T = 0.0244$ sec. But, $a = \frac{1}{T} = 40.984$, and DC gain is 2. Thus $\frac{K}{a} = 2$. Hence, $K = 81.967$.

Thus,

\[G(s) = \frac{81.967}{s + 40.984}\]

(b) This is a second-order system of the form: $G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$. We can estimate the percent overshoot and the settling time from the graph.

\[
\%OS = \left(\frac{13.82 - 11.03}{11.03}\right) \times 100 = 25.3\%
\]
\[
T_s = 2.62 \text{ sec}
\]

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We can now calculate $\zeta$ and $\omega_n$ from the given information.

$$\zeta = \frac{-\ln(\%\text{OS}/100)}{\sqrt{\pi^2 + \ln^2(\%\text{OS}/100)}} = 0.4$$

$$\omega_n = \frac{4}{\zeta T_s} = 3.82$$

DC Gain $= 11.03$. Therefore, $\frac{K}{\omega_n^2} = 11.03$. Hence, $K = 160.95$. Substituting all values, we get

$$G(s) = \frac{160.95}{s^2 + 3.056s + 14.59}$$

(c) This is a second-order system of the form: $G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$. We can estimate the percent overshoot and the peak time from the graph.

%OS $= \frac{(1.4 - 1.0)}{1.0} \times 100 = 40\%$

$T_p = 4$

We can now calculate $\zeta$ and $\omega_n$ from the given information.

$$\zeta = \frac{-\ln(\%\text{OS}/100)}{\sqrt{\pi^2 + \ln^2(\%\text{OS}/100)}} = 0.28$$

$$\omega_n = \frac{\pi}{T_p\sqrt{1 - \zeta^2}} = 0.818$$

DC Gain $= 1.0$. Therefore, $\frac{K}{\omega_n^2} = 1.0$. Hence, $K = 0.669$. Substituting all values, we get

$$G(s) = \frac{0.669}{s^2 + 0.458s + 0.669}$$

4. **State Space Analysis**

A system is represented by the state and output equations that follow. Without solving the state equation, find the characteristic equation and the poles of the system.

$$\dot{x} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 6 & 5 \\ 1 & 4 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} x$$

**Solution:**

$$sI - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 3 \\ 0 & 6 & 5 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} s & -2 & -3 \\ 0 & s - 6 & -5 \\ -1 & -4 & s - 2 \end{bmatrix}$$

Characteristic Equation: $\det (sI - A) = s^3 - 8s^2 - 11s + 8$

Factoring yields poles: $9.111, 0.5338$ and $-1.6448$
5. **Block Diagram To Transfer Function**

Reduce the system shown below to a single transfer function, \( T(s) = C(s)/R(s) \).

![Block Diagram](image)

**Solution:** Push \( G_2(s) \) to the left past the summing junction.

![Block Diagram](image)

Collapse the summing junctions and add the parallel transfer functions.
Push $G_1(s)G_2(s) + G_3(s)$ to the right past the summing junction.

Collapse the summing junctions and add feedback paths.

Applying the feedback formula,

$$T(s) = \frac{G_3(s) + G_1(s)G_2(s)}{1 + H(s)[G_3(s) + G_1(s)G_2(s)] + G_2(s)G_4(s)}$$

6. **Block Diagram To Transfer Function**
For the system shown below, find the poles of the closed-loop transfer function, $T(s) = C(s)/R(s)$. 

![Block Diagram](image)
**Solution:** Push $2s$ to the left past the pickoff point and combine the parallel combination of $2$ and $1/s$.

![Diagram 1](image1)

Push $(2s + 1)/s$ to the right past the summing junction and combine summing junctions.

![Diagram 2](image2)

Hence,

$$T(s) = \frac{2(2s + 1)}{1 + 2(2s + 1)H_{eq}(s)}$$

where $H_{eq} = 1 + \frac{s}{2s + 1} + \frac{5}{2s}$.

$$T(s) = \frac{4s^2 + 2s}{6s^2 + 13s + 5}$$

7. **Finding Constants To Meet Design Specifications**

For the system shown below, find the values of $K_1$ and $K_2$ to yield a peak time of 1.5 seconds and a settling time of 3.2 seconds for the closed loop system’s step response.

![Diagram 3](image3)
Solution: The closed-loop transfer function is given by:

\[ T(s) = \frac{10K_1}{s^2 + (10K_2 + 2)s + 10K_1} \]

We know, \( \zeta \omega_n = \frac{4}{T_s} = 1.25 \). And \( \omega_n \sqrt{1 - \zeta^2} = \frac{\pi}{T_p} = 2.09 \). Therefore, poles are at \(-\zeta \omega_n \pm \omega_n \sqrt{1 - \zeta^2} = -1.25 \pm j2.09\). Hence, \( \omega_n = \sqrt{1.25^2 + 2.09^2} = \sqrt{10K_1} \) and \((10K_2 + 2)/2 = 1.25\). Therefore, \( K_1 = 0.593 \) and \( K_2 = 0.05 \).

8. Signal-Flow Graphs
Draw a signal-flow graph for the following equation.

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2 & -4 & -6
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} r(t)
\]

\[ y = \begin{bmatrix}
1 & 1 & 0
\end{bmatrix} x \]

Solution:

\( \dot{x}_1 = x_2 \)
\( \dot{x}_2 = x_3 \)
\( \dot{x}_3 = -2x_1 - 4x_2 - 6x_3 + r \)
\( y = x_1 + x_2 \)

9. Signal-Flow Graphs
Using Mason’s rule, find the transfer function, \( T(s) = C(s)/R(s) \), for the system represented by the following figure.
Solution:

Closed-loop gains: \( G_2G_4G_6G_7H_3; G_2G_5G_6G_7H_3; G_3G_4G_6G_7H_3; G_6H_1; G_7H_2 \)

Forward-path gains: \( T_1 = G_1G_2G_4G_6G_7; T_2 = G_1G_2G_5G_6G_7; T_3 = G_1G_3G_4G_6G_7; T_4 = G_1G_3G_5G_6G_7 \)

Nontouching loops 2 at a time: \( G_6H_1G_7H_2 \)

\[ \Delta = 1 - [H_3G_6G_7(G_2G_4 + G_2G_5 + G_3G_5) + G_6H_1 + G_7H_2] + [G_6H_1G_7H_2] \]

\[ \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1 \]

\[ T(s) = \frac{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3 + T_4\Delta_4}{\Delta} \]

\[ T(s) = \frac{G_1G_2G_4G_6G_7 + G_1G_2G_5G_6G_7 + G_1G_3G_4G_6G_7 + G_1G_3G_5G_6G_7}{1 - H_3G_6G_7(G_2G_4 + G_2G_5 + G_3G_4 + G_3G_5) - G_6H_1 - G_7H_2 + G_6H_1G_7H_2} \]

10. State Space Representation and Signal-Flow Graphs

Represent the system shown below in state space form and draw its signal-flow graph.

\[ G(s) = \frac{s + 3}{s^2 + 2s + 7} \]

Solution: Writing the state equations,

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -7x_1 - 2x_2 + r \]
\[ y = 3x_1 + x_2 \]

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \]
\[ y = \begin{bmatrix} 3 & 1 \end{bmatrix} x \]
11. **Canonical Forms**

Represent the system given in Problem 10 in controller canonical form and observer canonical form.

**Solution:**

Controllable canonical form:
\[
\dot{x} = \begin{bmatrix} -2 & -7 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \\
y = \begin{bmatrix} 1 & 3 \end{bmatrix} x
\]

Observable canonical form:
\[
\dot{x} = \begin{bmatrix} -2 & 1 \\ -7 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]