8. For the system in Problem 35, compute the following steady-state errors:

(a) to a unit-step reference input;
(b) to a unit-ramp reference input;
(c) to a unit-step disturbance input;
(d) for a unit-ramp disturbance input.

(e) Verify your answers to parts (a) to (d) using MATLAB. Note that a ramp response can be generated as the step response of a system modified by an added integrator at the reference input.

Solution:

(a)

\[
\Omega_r(t) = u(t) \implies \Omega_r(s) = \frac{1}{s}
\]

\[
e_{ss} = \lim_{s \to 0} s\Omega_r(s) \frac{1}{1 + G(s)}
\]

\[
= \lim_{s \to 0} s \left( \frac{1}{s} \frac{1}{1 + (k_p + \frac{k_I}{s})(\frac{600}{s+60})} \right)
\]

\[
= 0
\]

(b)

\[
\Omega_r(t) = r(t) \implies \Omega_r(s) = \frac{1}{s^2}
\]

\[
e_{ss} = \lim_{s \to 0} s \frac{1}{s^2} \left( \frac{1}{1 + (k_p + \frac{k_I}{s})(\frac{600}{s+60})} \right)
\]

\[
= \frac{1}{10k_I}
\]

(c)

\[
e_{ss} = \lim_{s \to 0} [sW(s) \frac{1500}{600} \frac{600}{s+60} \frac{600}{1 + \frac{600}{s+60}(k_p + \frac{k_I}{s})}]
\]

\[
W(s) = \frac{1}{s}
\]

\[
e_{ss} = \lim_{s \to 0} [s \frac{1500}{s} \frac{600}{s+60} \frac{600}{1 + \frac{600}{s+60}(k_p + \frac{k_I}{s})}]
\]

\[
= 0
\]
\[ W(s) = \frac{1}{s^2} \]

\[ \varepsilon_{ss} = \lim_{s \to 0} \left( s^2 \frac{1500}{600} \frac{600}{s + 60} \left( \frac{600}{s + 60} \left( k_p + \frac{k_I}{s} \right) \right) \right) \]

\[ = \frac{15}{6} \frac{1}{k_I} = 2.5 \frac{1}{k_I} \]

See attached transient responses.
9. Consider the system shown in Fig. 4.33. Show that the system is type 1 and compute the $K_v$.

![Figure 4.33: Control system for Problem 9](image)

Solution:

The system has unity feedback with one pole at $s = 0$ and is thus Type 1 with $K_v = \lim_{s \to 0} sG(s) = K_b$. 
11. Consider the system shown in Fig. 4.35, where

\[ D(s) = K \frac{(s + \alpha)^2}{s^2 + \omega_n^2}. \]

![Control system diagram](image)

Figure 4.35: Control system for Problem 11

(a) Prove that if the system is stable, it is capable of tracking a sinusoidal reference input \( r = \sin \omega_n t \) with zero steady-state error. (Look at the transfer function from \( R \) to \( E \) and consider the gain at \( \omega_n \).

(b) Use Routh’s criteria to find the range of \( K \) such that the closed-loop system remains stable if \( \omega_n = 1 \) and \( \alpha = 0.25 \).

**Solution:**

(a)

\[
D(s)G(s) = \frac{K(s + \alpha)^2}{(s^2 + \omega_n^2)s(s + 1)}
\]

\[
E(s) = \frac{1}{1 + DG}
\]

\[
= \frac{s(s + 1)(s^2 + \alpha^2)}{(s^2 + \omega_n^2)s(s + 1) + K(s + \alpha)^2}
\]

The gain of this transfer function is zero at \( s = \pm j \omega_n \) and we expect the error to be zero if \( R \) is a sinusoid at that frequency. More formally, let \( R(s) = \frac{\omega_n}{s^2 + \omega_n^2} \) then

\[
E(s) = \frac{s(s + 1)(s^2 + \omega_n^2)}{(s^2 + \omega_n^2)s(s + 1) + K(s + \alpha)^2} \frac{\omega_n}{s^2 + \omega_n^2}
\]

Assuming the (closed-loop) system is stable, then if \( \omega_n = \omega_o \) \( E(s) \) has a pole on the imaginary axis and the FVT does not apply. The final error will NOT be zero in this case. However, if \( \omega_n = \omega_o \) we can use the FVT and

\[
e_{ss} = \lim_{s \to 0} sE(s) = 0
\]
12. Consider the system shown in Fig. 4.36 which represents control of the angle of a pendulum which has no damping.

\[ Y = \frac{1}{s^2}(W + D(R - Y) - KY) \]

\[ Y\left(\frac{s^2 + D + K}{s^2}\right) = \frac{W + DR}{s^2} \]

\[ Y = \frac{D}{s^2 + D + K} R + \frac{1}{s^2 + D + K} W \]

\[ E(s) = R(s) - Y(s) = \frac{-D + s^2 + D + K}{s^2 + D + K} R(s) \]

\[ = \frac{s^2 + K}{s^2 + D + K} R(s) \]
for constant steady-state error to a ramp,

\[
\lim_{s \to 0} s\left(\frac{s^2 + K}{s^2 + D + K}\right) = \text{constant}
\]

\[
\lim_{s \to 0} sD(s) = \text{constant}
\]

\[D(s)\text{ must have a pole at the origin.}\]

(b)

\[
Y(s) = \frac{1}{s^2 + D(s) + K}W(s)
\]

\[
\lim_{s \to 0} s\left(\frac{1}{s^2 + D(s) + K}\right) = 0
\]

iff

\[
\lim_{s \to 0} s^{\ell-1}D(s) = \infty
\]

iff \(\ell = 1\) since \(D(s)\) has a pole at the origin. Therefore system will reject step disturbances.

(c) For PI-controller,

\[
D(s) = (k_p + \frac{k_I}{s})
\]

\[
\frac{Y(s)}{R(s)} = \frac{D(s)}{s^2 + D(s) + K} = \frac{k_p s + k_I}{s^2 + (\frac{k_p s + k_I}{s}) + K}
\]

\[
= \frac{k_p s + k_I}{s^3 + (k_p s + k_I) + K s}
\]

Because there is no term in \(s^2\) this characteristic equation must have at least one pole in the right half-plane. Try PID the controller, \(D(s) = (k_p + k_D s + \frac{k_I}{s})\)

\[
\frac{Y(s)}{R(s)} = \frac{k_D s^2 + k_p s + k_I}{s^3 + (k_D s^2 + k_p s + k_I) + K s}
\]

\[
= \frac{k_D s^2 + k_p s + k_I}{s^3 + k_D s^2 + (k_p + K)s + k_I}
\]

Routh’s test on the characteristic equation is:

\[
\begin{array}{ccc}
s^3 : & 1 & K + k_p \\
s^2 : & k_D & k_I \\
s : & k_D(k_p + K) - k_I & 0 \\
s^0 : & k_D & k_I & 0
\end{array}
\]
17. A controller for a satellite attitude control with transfer function \( G = \frac{1}{s^2} \) has been designed with a unity feedback structure and has the transfer function \( D(s) = \frac{10(s + 2)}{s + 5} \).

(a) Find the system type for reference tracking and the corresponding error constant for this system.

(b) If a disturbance torque adds to the control so that the input to the process is \( u + w \), what is the system type and corresponding error constant with respect to disturbance rejection?

Solution:

(a)

\[ K_p = \lim_{s \to 0} D(s)G(s) = \infty \]

\[ e_{ss} = \frac{1}{1 + K_p} = 0. \]

\[ K_v = \lim_{s \to 0} sD(s)G(s) = \infty \]

\[ e_{ss} = \frac{1}{K_v} = 0. \]

\[ K_a = \lim_{s \to 0} s^2D(s)G(s) = 4 \]

\[ e_{ss} = \frac{1}{K_a} = 0.25. \]

(b) For the disturbance input, the error is

\[ \frac{E(s)}{W(s)} = -\frac{G}{1 + GD} = -\frac{s + 5}{s^2(s + 5) + 10(s + 2)} \]

The steady-state error to a step is thus \( e_{ss} = 0.25 = \frac{1}{1 + K_p} \). Therefore,

\[ K_p = 3 \]
CHAPTER 5. THE ROOT-LOCUS DESIGN METHOD

Solution:

(a) \( K = 1/\tau; a = s; b = 1 \)

(b) \( K = c; a = s^2 + 1; b = s + 1 \)

(c) Part (c)
   i. \( K = AT; a = (s + c)^3; b = s + 1/T \)
   ii. \( K = AT; a = (s + c)^3 + A; b = s \)
   iii. The parameter \( c \) enters the equation in a nonlinear way and a standard root locus does not apply. However, using a polynomial solver, the roots can be plotted versus \( c \).

(d) Part (d)
   i. \( K = k_p A \tau; a = s(s + 1/\tau)d(s) + k_I(s + 1/\tau)c(s) + \frac{k_D}{\tau}s^2 Ac(s); \quad b = s(s + 1/\tau)c(s) \)
   ii. \( K = Ak_I; a = s(s + 1/\tau)d(s) + Ak_p s(s + 1/\tau) + \frac{k_D}{\tau}s^2 Ac(s); \quad b = s(s + 1/\tau)c(s) \)
   iii. \( K = \frac{Ak_D}{\tau}; a = s(s + 1/\tau)d(s) + Ak_p s(s + 1/\tau)c(s) + Ak_I(s + 1/\tau)c(s); \quad b = s^2 c(s) \)
   iv. \( K = 1/\tau; a = s^2 d(s) + k_p A s^2 c(s) + k_I A s c(s); \quad b = s d(s) + k_p A s c(s) + k_I A c(s) + k_D s^2 A c(s) \)

Problems and solutions for Section 5.2

2. Roughly sketch the root loci for the pole-zero maps as shown in Fig. 5.62. Show your estimates of the center and angles of the asymptotes, a rough evaluation of arrival and departure angles for complex poles and zeros, and the loci for positive values of the parameter \( K \). Each pole-zero map is from a characteristic equation of the form

\[ 1 + K \frac{b(s)}{a(s)} = 0, \]

where the roots of the numerator \( b(s) \) are shown as small circles \( o \) and the roots of the denominator \( a(s) \) are shown as \( \times \)s on the \( s \)-plane. Note that in Fig. 5.62(c), there are two poles at the origin.

Solution:

(a) \( a(s) = s^2 + s; b(s) = s + 1 \)

Breakin(s) -3.43; Breakaway(s) -0.586

(b) \( a(s) = s^2 + 0.2s + 1; b(s) = s + 1 \)
Figure 5.62: Pole-zero maps from Figure 5.62

Angle of departure: 135.7
Breakin(s) -4.97
(c) $a(s) = s^2; \ b(s) = (s + 1)$
Breakin(s) -2
(d) $a(s) = s^2 + 5s + 6; \ b(s) = s^2 + s$
Breakin(s) -2.37
Breakaway(s) -0.634
(e) $a(s) = s^3 + 3s^2 + 4s - 8$
Center of asymptotes -1
Angles of asymptotes ±60, 180
Angle of departure: -56.3
(f) $a(s) = s^3 + 3s^2 + s - 5; \ b(s) = s + 1$
Center of asymptotes -0.667
Angles of asymptotes ±60, -180
Angle of departure: -90
Breakin(s) -2.06
Breakaway(s) 0.503

3. For the characteristic equation

\[ 1 + \frac{K}{s(s + 1)(s + 5)} = 0 : \]

(a) Draw the real-axis segments of the corresponding root locus.
(b) Sketch the asymptotes of the locus for \( K \to \infty \).
(c) For what value of \( K \) are the roots on the imaginary axis?
(d) Verify your sketch with a MATLAB plot.

**Solution:**

(a) The real axis segments are \( 0 > \sigma > -1; -5 > \sigma \)
(b) \( \alpha = -6/3 = -2; \phi_i = \pm 60, 180 \)
(c) \( K_o = 30 \)

Solution for Problem 5.3

4. *Real poles and zeros.* Sketch the root locus with respect to \( K \) for the equation \( 1 + KL(s) = 0 \) and the following choices for \( L(s) \). Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) \( L(s) = \frac{1}{s(s + 1)(s + 5)(s + 10)} \)
(b) \( L(s) = \frac{s + 2}{s(s + 1)(s + 5)(s + 10)} \)
(c) \( L(s) = \frac{(s + 2)(s + 6)}{s(s + 1)(s + 5)(s + 10)} \)
(d) \[ L(s) = \frac{(s + 2)(s + 4)}{s(s + 1)(s + 5)(s + 10)} \]

**Solution:**

All the root locus plots are displayed at the end of the solution set for this problem.

(a) \( \alpha = -4; \phi_i = \pm 45; \pm 135; \omega_o = 1.77 \)
(b) \( \alpha = -4.67; \phi_i = \pm 60; \pm 180; \omega_o = 5.98 \)
(c) \( \alpha = -4; \phi_i = \pm 90; \omega_o \rightarrow \text{none} \)
(d) \( \alpha = -5; \phi_i = \pm 90; \omega_o \rightarrow \text{none} \)

---

5. **Complex poles and zeros** Sketch the root locus with respect to \( K \) for the equation \( 1 + KL(s) = 0 \) and the following choices for \( L(s) \). Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) \[ L(s) = \frac{1}{s^2 + 3s + 10} \]
(b) \[ L(s) = \frac{1}{s(s^2 + 3s + 10)} \]
(c) \[ L(s) = \frac{(s^2 + 2s + 8)}{s(s^2 + 2s + 10)} \]

(d) \[ L(s) = \frac{(s^2 + 12)}{s(s^2 + 2s + 10)} \]

(e) \[ L(s) = \frac{(s^2 + 1)}{s(s^2 + 4)} \]

(f) \[ L(s) = \frac{(s^2 + 4)}{s(s^2 + 1)} \]

Solution:

All the root locus plots are displayed at the end of the solution set for this problem.

(a) \[ \alpha = -3; \phi_i = \pm 90; \theta_d = \pm 90 \text{ } \omega_o \rightarrow \text{none} \]

(b) \[ \alpha = -3; \phi_i = \pm 60, \pm 180; \theta_d = \pm 28.3 \text{ } \omega_o = 3.16 \]

(c) \[ \alpha = -2; \phi_i = \pm 180; \theta_d = \pm 161.6; \theta_a = \pm 200.7; \omega_o \rightarrow \text{none} \]

(d) \[ \alpha = -2; \phi_i = \pm 180; \theta_d = \pm 18.4; \theta_a = \pm 16.8; \omega_o \rightarrow \text{none} \]

(e) \[ \alpha = 0; \phi_i = \pm 180; \theta_d = \pm 180; \theta_a = \pm 180; \omega_o \rightarrow \text{none} \]

(f) \[ \alpha = 0; \phi_i = \pm 180; \theta_d = 0; \theta_a = 0; \omega_o \rightarrow \text{none} \]

Solution to Problem 5.5
6. Multiple poles at the origin  Sketch the root locus with respect to $K$ for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \frac{1}{s^2(s + 8)}$

(b) $L(s) = \frac{1}{s^3(s + 8)}$

(c) $L(s) = \frac{1}{s^4(s + 8)}$

(d) $L(s) = \frac{(s + 3)}{s^2(s + 8)}$

(e) $L(s) = \frac{(s + 3)}{s^3(s + 4)}$

(f) $L(s) = \frac{(s + 1)^2}{s^3(s + 4)}$

(g) $L(s) = \frac{(s + 1)^2}{s^3(s + 10)^2}$

Solution:

All the root locus plots are displayed at the end of the solution set for this problem.

(a) $\alpha = -2.67; \phi_i = \pm 60; \pm 180; w_0^- > none$

(b) $\alpha = -2; \phi_i = \pm 45; \pm 135; w_0^- > none$

(c) $\alpha = -1.6; \phi_i = \pm 36; \pm 108; w_0^- > none$

(d) $\alpha = -2.5; \phi_i = \pm 90; w_0^- > none$

(e) $\alpha = -0.33; \phi_i = \pm 60; \pm 180; w_0^- > none$

(f) $\alpha = -3; \phi_i = \pm 90; w_0 = \pm 1.414$

(g) $\alpha = -6; \phi_i = \pm 60; 180; w_0 = \pm 1.31; \pm 7.63$
Solution for Problem 5.6

7. Mixed real and complex poles Sketch the root locus with respect to \( K \) for the equation \( 1 + KL(s) = 0 \) and the following choices for \( L(s) \). Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) \( L(s) = \frac{(s + 2)}{s(s + 10)(s^2 + 2s + 2)} \)

(b) \( L(s) = \frac{(s + 2)}{s^2(s + 10)(s^2 + 6s + 25)} \)

(c) \( L(s) = \frac{(s + 2)^2}{s^2(s + 10)(s^2 + 6s + 25)} \)

(d) \( L(s) = \frac{(s + 2)(s^2 + 4s + 68)}{s^2(s + 10)(s^2 + 4s + 85)} \)

(e) \( L(s) = \frac{(s + 1)^2 + 1}{s^2(s + 2)(s + 3)} \)

Solution:

All the plots are attached at the end of the solution set.

(a) \( \alpha = -3.33; \ \phi_l = \pm 60; \ \pm 180; \ \phi_0 = \pm 2.32; \ \theta_d = \pm 6.34 \)

(b) \( \alpha = -3.5; \ \phi_l = \pm 45; \ \pm 135; \ \phi_0 \rightarrow \text{none}; \ \theta_d = \pm 103.5 \)
(c) $\alpha = -4; \phi_i = \pm 60; \pm 180; w_0 = \pm 6.41; \quad \theta_d = \pm 14.6$

(d) $\alpha = -4; \phi_i = \pm 90; w_0 > \text{none}; \quad \theta_d = \pm 106; \quad \theta_a = \pm 253.4$

(e) $\alpha = -1.5; \phi_i = \pm 90; w_0 > \text{none}; \quad \theta_b = \pm 71.6$

8. **Right half plane poles and zeros** Sketch the root locus with respect to $K$ for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \frac{s + 2}{s + 10} \frac{1}{s^2 - 1}$: The model for a case of magnetic levitation with lead compensation.

(b) $L(s) = \frac{s + 2}{s(s + 10)} \frac{1}{(s^2 - 1)}$: The magnetic levitation system with integral control and lead compensation.

(c) $L(s) = \frac{s - 1}{s^2}$

(d) $L(s) = \frac{s^2 + 2s + 1}{s(s + 20)^2(s^2 - 2s + 2)}$. What is the largest value that can be obtained for the damping ratio of the stable complex roots on this locus?