Example 1. Consider the signal $x(t)$ as shown below

1. Find the FT $X(j\omega)$ of $x(t)$

2. Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$

3. Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$

Solution:

1. Not that $x(t) = x_1(t) * x_1(t)$ where

   $$x_1(t) = \begin{cases} 1, & |t| \leq \frac{1}{2} \\ 0, & \text{o.w.} \end{cases}$$

   Also,

   $$X_1(j\omega) = \int_{-1/2}^{1/2} e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \bigg|_{t=-1/2}^{1/2} = \frac{e^{-j\omega/2} - e^{j\omega/2}}{-j\omega} = \frac{2e^{j\omega/2} - e^{-j\omega/2}}{2j} = \frac{\sin \left( \frac{\omega}{2} \right)}{\omega}$$

   Using the convolution property, we get

   $$X(j\omega) = X_1(j\omega)X_1(j\omega) = \left( \frac{2\sin \left( \frac{\omega}{2} \right)}{\omega} \right)^2$$

2. $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - 4k)$$
3. One possible choice for $g(t)$ is

Example 2. Find the DTFT of $na^{n+1}u[n + 1]$, $|a| < 1$.

Solution:
Note that

$$na^{n+1}u[n + 1] = (n + 1)a^{n+1}u[n + 1] - a^{n+1}u[n + 1]$$

(This is just zero rewritten). Further,

$$a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

The DTFT of the second term can be obtained by the time–shift property

$$a^{n+1}u[n+1] \leftrightarrow \frac{e^{j\omega}}{1 - ae^{-j\omega}}$$

The DTFT of the first term can be obtained using the frequency differentiation property

$$na^n u[n] \leftrightarrow j \frac{d}{d\omega} \left( \frac{1}{1 - ae^{-j\omega}} \right) = j(-1) \frac{-ae^{-j\omega}(-j)}{(1 - ae^{-j\omega})^2} = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$$

Hence,

$$(n + 1)a^{n+1}u[n + 1] \leftrightarrow \frac{ae^{-j\omega}e^{j\omega}}{(1 - ae^{-j\omega})^2} = \frac{a}{(1 - ae^{-j\omega})^2}$$

Combining the two pieces we get

$$na^{n+1}u[n + 1] \leftrightarrow \frac{a}{(1 - ae^{-j\omega})^2} - \frac{e^{j\omega}}{1 - ae^{-j\omega}}$$
Example 3. Find the DTFT of \( x[n] \equiv 1 \)

Solution:
Note that if \( \tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \) then
\[
\tilde{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega - 2\pi k) e^{j\omega n} \, d\omega = \frac{1}{2\pi} \forall n
\]
Hence, if \( x[n] \equiv 1 \), we have
\[
X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)
\]

Example 4. Consider the difference equation
\[
y[n] - 1.2y[n-1] + 0.36y[n-2] = x[n] + x[n-1]
\]
1. Find the frequency response \( H(e^{j\omega}) \)
2. Use \( H(e^{j\omega}) \) to calculate the impulse response.

Solution:
1. 
\[
h[n] - 1.2h[n-1] + 0.36h[n-2] = \delta[n] + \delta[n-1]
\]
taking the FT we have
\[
H(e^{j\omega}) \left( 1 - 1.2e^{-j\omega} + 0.36e^{-j2\omega} \right) = (1 + e^{-j\omega})
\]
so that
\[
H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - 1.2e^{-j\omega} + 0.36e^{-j2\omega}}
\]
2. Note that \( z^2 - 1.2z + 0.36 = (z - 0.6)^2 \) so we can use partial fraction expansion:
\[
H(e^{-j\omega}) = \frac{A}{1 - 0.6e^{-j\omega}} + \frac{B}{(1 - 0.6e^{-j\omega})^2}
\]
Hence,
\[
A - 0.6Ae^{-j\omega} + B = 1 + e^{-j\omega} \implies A = \frac{5}{3}, B = \frac{8}{3}
\]
plugging in these values and taking the inverse FT we get
\[
h[n] = \frac{5}{3}(0.6)^n u[n] + \frac{8}{3} (n+1)(0.6)^n u[n] = (0.6^n + \frac{8}{3} n(0.6)^n) u[n]
\]

Example 5. Let \( S(t) \) be a real-valued signal for which \( S(j\omega) = 0 \) when \( |\omega| > 1000\pi \). Amplitude modulation is performed to produce the signal:
\[
r(t) = s(t) \sin(1000\pi t)
\]
and the demodulation scheme depicted below is applied to \( r(t) \) at the receiver end. Determine \( y(t) \) assuming that the ideal lowpass filter has a cutoff frequency of 1000\pi and a passband gain of 1.
Solution:
Let us denote by $x(t)$ the signal that is filtered by the ideal low pass filter

$$x(t) = s(t) \sin(1000\pi t) \cos(1000\pi t) = \frac{1}{2} s(t) \sin(2000\pi t)$$

Taking the FT of $x(t)$ we get

$$X(j\omega) = \frac{1}{4j} (S(j(\omega - 2000\pi)) - S(j(\omega + 2000\pi)))$$

(since $\sin \theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ and we know that $e^{j\theta}g(t) \longleftrightarrow G(\omega - \theta)$.) Since $s(t)$ is bandlimited to $1000\pi$ it is easy to conclude that $X(j\omega)$ is non-zero only in the range $[-3000\pi, -1000\pi]$ and $[1000\pi, 3000\pi]$. Because the low-pass filter passes frequency in the range from $-1000\pi$ to $1000\pi$, $Y(j\omega) = 0$, and thus $y(t) = 0$. 