Minimum Spanning Trees

**Problem:** Given a set of places and distances between them (assume always positive), find a set of connecting roads of minimum total length that allows travel between any two.

- The routes you get will not necessarily be shortest paths.
- Easy to see that such a set of connecting roads and places must form a tree, because removing one road in a cycle still allows all to be reached.
Minimum Spanning Trees by Prim’s Algorithm

- Idea is to grow a tree starting from an arbitrary node.
- At each step, add the shortest edge connecting some node already in the tree to one that isn’t yet.
- Why must this work?

PriorityQueue fringe;
For each node v { v.dist() = ∞; v.parent() = null; }
Choose an arbitrary starting node, s;
s.dist() = 0;
fringe = priority queue ordered by smallest .dist();
add all vertices to fringe;
while (! fringe.isEmpty()) {
    Vertex v = fringe.removeFirst();
    For each edge (v,w) {
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Minimum Spanning Trees by Kruskal’s Algorithm

• Observation: the shortest edge in a graph can always be part of a minimum spanning tree.

• In fact, if we have a bunch of subtrees of a MST, then the shortest edge that connects two of them can be part of a MST, combining the two subtrees into a bigger one.

• So,…

Create one (trivial) subtree for each node in the graph;
MST = {};

for each edge (v,w), in increasing order of weight {
    if ( (v,w) connects two different subtrees ) {
        Add (v,w) to MST;
        Combine the two subtrees into one;
    }
}
Recursive Depth-First Traversal

- Previously, we saw an iterative way to do depth-first traversal of a graph from a particular node.
- We are often interested in traversing all nodes of a graph, so we can repeat the procedure as long as there are unmarked nodes.
- Recursive solution is also simple:

```c
void traverse (Graph G) {
    for (v ∈ nodes of G) {
        traverse (G, v);
    }
}

void traverse (Graph G, Node v) {
    if (v is unmarked) {
        mark (v);
        visit v;
        for (Edge (v, w) ∈ G)
            traverse (G, w);
    }
}
```
Another Take on Topological Sort

- Observation: if we do a depth-first traversal on a DAG whose edges are reversed, and execute the recursive `traverse` procedure, we finish executing `traverse(G,v)` in proper topologically sorted order.

```java
void topologicalSort (Graph G) {
    for (v ∈ nodes of G) {  
        traverse (G, v);
    }
}

void traverse (Graph G, Node v) {  
    if (v is unmarked) {  
        mark (v);
        for (Edge (w, v) ∈ G)  
            traverse (G, w);
        add v to the result list;
    }
}
```


**Union Find**

- Kruskal’s algorithm required that we have a set of sets of nodes with two operations:
  - *Find* which of the sets a given node belongs to.
  - Replace two sets with their *union*, reassigning all the nodes in the two original sets to this union.

- Obvious thing to do is to store a set number in each node, making finds fast.

- Union requires changing the set number in one of the two sets being merged; the smaller is better choice.

- This means an individual union can take $\Theta(N)$ time.

- Can union be fast?
A Clever Trick

- Let’s choose to represent a set of nodes by one arbitrary representative node in that set.
- Let every node contain a pointer to another node in the same set.
- Arrange for each pointer to represent the parent of a node in a tree that has the representative node as its root.
- To find what set a node is in, follow parent pointers.
- To union two such trees, make one root point to the other (choose the root of the higher tree as the union representative).
Path Compression

• This makes unioning really fast, but the find operation potentially slow ($\Omega(lg N)$).

• So use the following trick: whenever we do a find operation, compress the path to the root, so that subsequent finds will be faster.

• That is, make each of the nodes in the path point directly to the root.

• Now union is very fast, and sequence of unions and finds each have very, very nearly constant amortized time.

• Example: find 'g' in last tree (result of compression on right):

![Diagram of tree before and after path compression]

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