EE40-FA09 HW Solutions

In this solution set the prime symbol indicates derivative: \( v' = \frac{dv}{dt} \)

**Problem 1:**
\( V(t) = V_0 \left( \exp(-t/RC) \right) \)
\(-\) \( V(\infty)/V_0 = 1/e = 37\% \)
\( W(t) = \frac{1}{2} C V^2(t) \)
\(-\) \( W(t)/W_0 = V^2(t)/V_0^2 \)
\(-\) \( W(\infty)/W_0 = 1/e^2 = 14\% \)

**Problem 2:**
\( a) \) Since the capacitor voltages cannot change instantaneously, at \( t=0^+ \):
\(-\) \( I = 100V/100k\Omega = 1mA \)
\( b) \) \(-\) \( V_1 + Ri + V_2 = 0 \)
\(-\) \( -V_1' + Ri' + V_2' = 0 \)
\(-\) \( V_1' = -i/C_1; V_2' = i/C_2 \)
\(-\) \( i(1/C_1 + 1/C_2) + Ri' = 0 \)
\( c) \) from \( b) \) \( RC_1C_2/(C_1+C_2) = 50ms \)
\( d) \) The solution is of the form \( K_1 \exp(-t/\tau) \). We find \( K_1 \) and \( \tau \) from parts \( a) \) and \( c) \) respectively.
\(-\) \( i(t) = 1mA \exp(-t/50ms) \)
\( e) \) \( C_1 V_1 = (C_2+C_1) V_t \)
\(-\) \( \rightarrow V_t = 50V \)

**Problem 3:**
\( a) \) We find the initial conditions and plug in to the known solution form:
\(-\) \[ \frac{V_{out}(t) - 10}{15k} + \frac{1}{\mu} \frac{dV_{out}(t)}{dt} = 0 \]
\(-\) \( V_{out}(0) = 10V \)
\(-\) \( V_{out}(t) = 10(1 - e^{-t/RC}) = 10(1 - e^{-t/0.0155}) \)
\( b) \) We again use the same strategy to find the time dependent form of \( V_{out} \) for \( t > 30ms \):
\(-\) \( V_{out}(0.03) = 10(1 - e^{-0.03/RC}) = 8.65V \)
\(-\) \( 10 = V_x(t) + 2V_x(t) \)
\(-\) \( V_x(t) = \frac{10}{3} V \)
\(-\) \[ V_{out}(t) = 2V_x + (8.65V - 2V_x)e^{-0.03/RC} = \frac{20}{3} + 1.98e^{-0.03/0.01} \]
\( c) \) The peak value is 8.65 V, and the asymptotic value is 6.66 V.
Problem 4:
Capacitor=Open Circuit; Inductor=short.
Hence
\[ V_C = -15 \text{V}; \]
\[ I_L = 5 \text{mA}; \]
\[ V_x = 5 \text{mA} \times 2 \text{k}\Omega = 10 \text{V} \]

Problem 5:
In steady state, the inductor acts as a short circuit. With the switch open, the steady-state current is:
\[ 100 \text{ V} / 100 \text{ \Omega} = 1 \text{ A} \]
With the switch closed, the current eventually approaches:
\[ i(\infty) = 100 \text{ V} / 25 \text{ \Omega} = 4 \text{ A} \]
For \( t > 0 \), the current has the form
\[ i(t) = K_1 + K_2 \exp(-R t / L) \]
where \( R = 25 \Omega \), because that is the resistance with the switch closed. Now, we have
\[ i(0^+) = i(0^-) = 1 \text{ A} = K_1 + K_2 \]
\[ i(\infty) = 4 \text{ A} = K_1 \]
Thus, we have \( K_2 = -3 \text{ A} \).
The current is
\[ i(t)=1A+3A(1-\exp(-t/80\text{ms})) \]
**Problem 6:**

The differential equation is:

\[ V_s = R i + L i' \]

\[ 10V \sin(300t) = 300 \, \Omega \, i + i'. \]

Substituting the particular solution

\[ A \cos(300t) + B \sin(300t), \]

we obtain

\[ 300(B-A) \sin(300t) = 10 \sin(300t) \]

\[ 300(B+A) \cos(300t) = 0 \]

Hence, \( B = -A = 1/60 \)

The particular solution is therefore:

\[ i_p(t) = -1/60(\cos(300t) + \sin(300t)) \]

and is such that \( i(0) = 1/60 \, A. \)

As a result, the homogeneous solution will need

\[ i_h(t) = -1/60 \exp(-300t) \]

so that \( i_h(0) + i_p(0) = 0. \)

The complete solution is then:

\[ i(t) = -1/60 \exp(-300t) - 1/60(\cos(300t) + \sin(300t)) \]

**Problem 7:**

The differential equation reads

\[ v(t) = R i + L i' \]

\[ v(t) = 10i = 10i + 2i' \]

Substituting the particular solution of the form \( A + Bt \) gives
10t=10A+10Bt+2B  
Solving this equation separately using the time-dependent and constant terms,  
10A+2B=0, B=1  
Hence A=-B/5=-0.2 and \(i_p(0)=-0.2\).  
The complementary solution is  
\(i_c(t) = K_1 \exp(-R \ t / L)\)  
Finally we apply the initial condition:  
\(i(0) = 0 = K_1 - 0.2\)  
so that \(i_n(t)=0.2\exp(-5t)\) and the complete solution is:  
\(i(t) = 0.2 \ A \ (\exp(-5 \ t) - 1) + 1A/s \ * \ t\)  

**Problem 8:**  
The differential equation can be obtaining by writing KVL and then differentiating LHS and RHS:  
\[d(10\cos(100t))/dt = L \ i'' + R \ i' + i/C = -1000\sin(100t)\]  
Substituting the particular solution:  
\(i(t)=A\cos(100t)+B\sin(100t)\)  
\(i'=100(-A\sin(100t) + B\cos(100t))\)  
\(i''=-10^4(A\cos(100t) + B\sin(100t))\)  
one obtains  
\(-10^4L(A\cos(100t) + B\sin(100t)) + 100R(-A\sin(100t) + B\cos(100t)) + 1/C(A\cos(100t) + B\sin(100t)) = -1000\sin(100t)\)  
Sdimplifying out the first and the third term, and dividing LHS and RHS by 1000  
\[5(-A\sin(100t)+B\cos(100t))=-\sin(100t)\] 
This results in B=0, A=1/5, for:  
\(i_p(t) = 0.2 \ A \ \cos(100 \ t)\)  
At t=0, this solution becomes \(i(0)=1/5\); furthermore \(i'(0)=0\).  
The homogeneous solution must be found by solving the roots of the characteristic polynomial:  
\[x^2+50x+10^4=0\]  
which gives  
\[x_1=-50+(50^2-4*10^4)/2=-50+j*(37500)^{0.5}\]  
\[x_2=-50+(50^2-4*10^4)/2=-50-j*(37500)^{0.5}\]  
Now note that we have
\[ \alpha = \frac{R}{2L} = 25 \text{ rad/s}; \quad \omega_0 = \frac{1}{\sqrt{LC}} = 100 \text{ rad/s} \]

Because we have \( \alpha < \omega_0 \), this circuit is underdamped. The natural frequency is given by Equation 4.76:

\[ \omega_n = \sqrt{\omega_0^2 - \alpha^2} = 96.82 \text{ rad/s} \]

Substituting in the initial conditions we find

\[ i_h(t) = \exp(-t^{*}25)(-0.2\cos(96.82t)-5.16\sin(96.82t)) \]

and the complete solution is:

\[ i(t) = 0.2 \text{ A} (\cos(100t) - \exp(25t)\cos(96.8t)) + 0.051 \text{ A} \exp(-25t)\sin(96.8t) \]

**Problem 9:**

a) (3 pts) Because the current through the inductor is continuous, \( i(0^+) = 0 \)

b) (3 pts) Use the voltage divider formula to find the initial voltage across the capacitor:

\[ \frac{80*15}{(9+15)} = 50 \text{ V} \]

c) (3 pts) Since the current in the inductor does not change, and the voltage on the capacitor does not change, there is 0 V drop across the resistor, and 50V of drop on the inductor. As a result,

\[ V = 100V - 50V = 50V = L \frac{di(0^+)}{dt} \]

\[ \frac{di(0^+)}{dt} = \frac{50V}{5m} = 10000 \text{ A/s} \]

d) (6 pts) The particular solution gives \( i(t) = 0 \). To find the homogeneous solution:

\[ \omega_0^2 = \frac{1}{LC} = 1e8 \]

\[ \alpha^2 = \left(\frac{R}{2L}\right)^2 = .64e8 \]

\[ \omega^2 > \alpha^2 \Rightarrow \text{Underdamped response (2pt)} \]

Thus we know the solution for \( i(t) \) is in the form

\[ i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \]

Using initial conditions from part a and b, we find the solution in the general form to be:

\[ i(t) = (1.67e^{-30000t} \sin 6000t) A \text{ for } t \geq 0. \]