EECS 151/251A Spring 2023

Digital Design and Integrated Circuits

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Lecture 6: Combinational Logic representations

Announcements

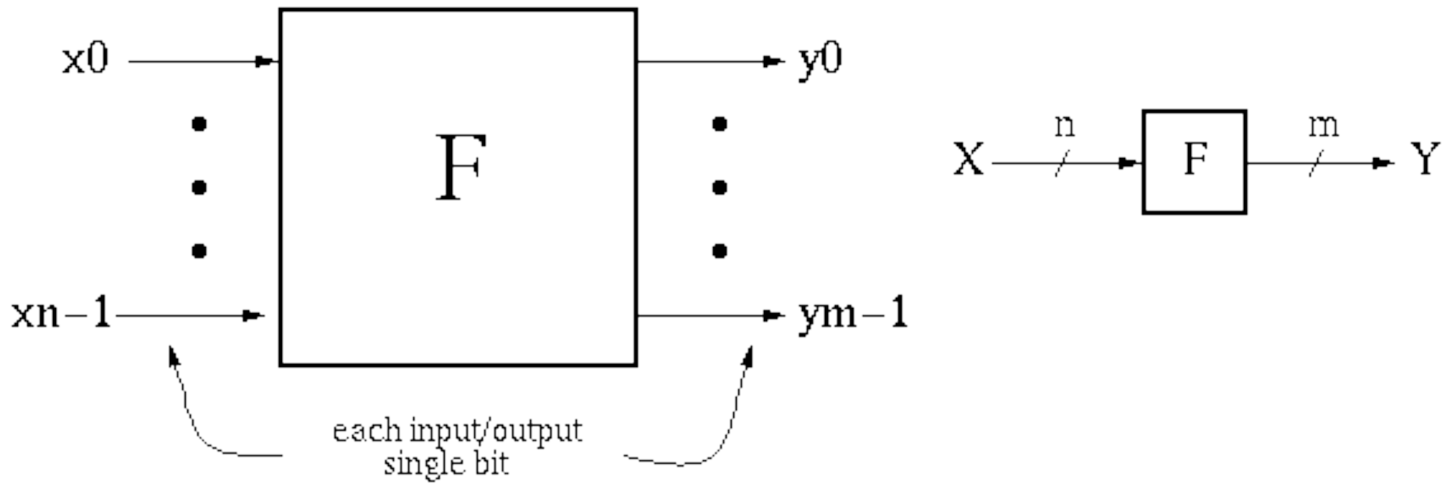
- PS 1 being graded, solution posted tomorrow
- **PS 2 due Monday**
- HW 3 out tomorrow.
- *Wawrzynek office hours moving to Thursday 12:30 starting next week.*

Outline

- *Three representations for combinational logic:*
  - *truth tables,*
  - *graphical (logic gates), and*
  - *algebraic equations*
- *Boolean Algebra*
- *Boolean Simplification*
- *Multi-level Logic,*
- *NAND/NOR*
- *XOR*

Representations of Combinational Logic

Combinational Logic (CL) Defined

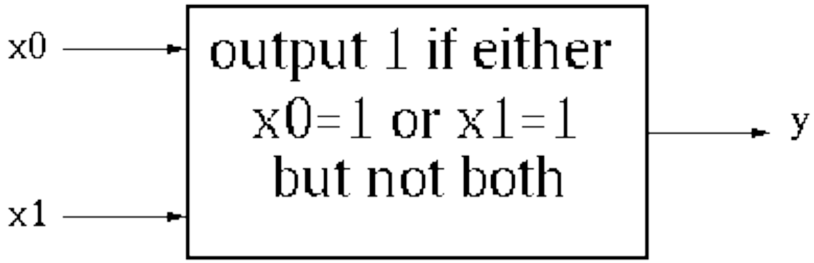


$y_i = f_i(x0 , ... , xn\text{-}1)$, where x, y take on values {0,1}.

Y is a function of only X, i.e., it is a "pure function".

- If we change X then Y will change immediately (well almost!).
  - There is an ***implementation dependent*** delay from X to Y.
  - Y is a function of nothing other than the current inputs values.

CL Block Example #1



Truth Table Description:

x0	x1	y
0	0	0
0	1	1
1	0	1
1	1	0

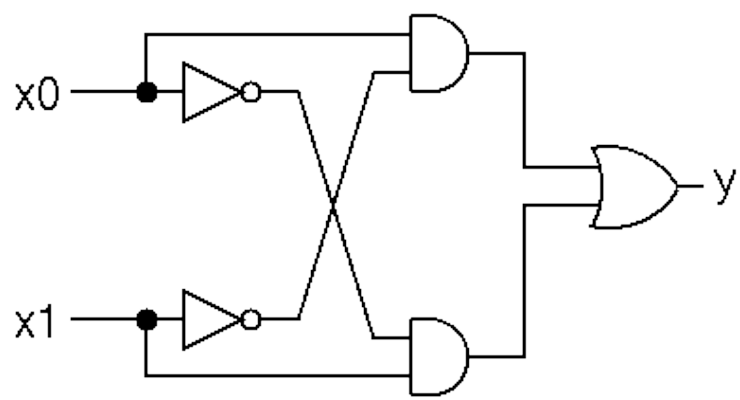
Boolean Equation:

$y_0 = (x_0 \text{ AND not}(x_1))$

$\text{OR } (\text{not}(x_0) \text{ AND } x_1)$

$y_0 = x_0 x_1' + x_0' x_1$

Gate Representation:

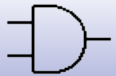


Boolean Algebra/Logic Circuits

❑ Why are they called "logic circuits"?

- Logic: The study of the principles of reasoning.
- The 19th Century Mathematician, George Boole, developed a math. system (algebra) involving logic, Boolean Algebra.
- His variables took on TRUE, FALSE
- Later Claude Shannon (father of information theory) showed (in his Master's thesis!) how to map Boolean Algebra to digital circuits:
- Primitive functions of Boolean Algebra:

a b	AND
0 0	0
0 1	0
1 0	0
1 1	1



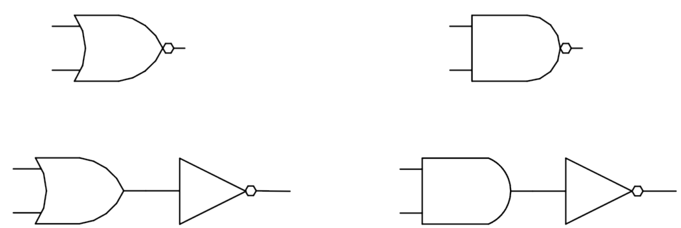
a b	OR
0 0	0
0 1	1
1 0	1
1 1	1

a	NOT
0	0
1	0

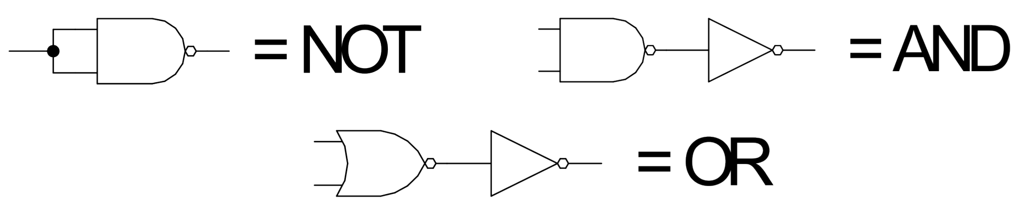
Other logic functions of 2 variables (x,y)

x y	f0	f1														
00	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
01	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
	0	AND		X		Y	⊕	OR	NOR	XNOR					NAND	1

Look at NOR and NAND:



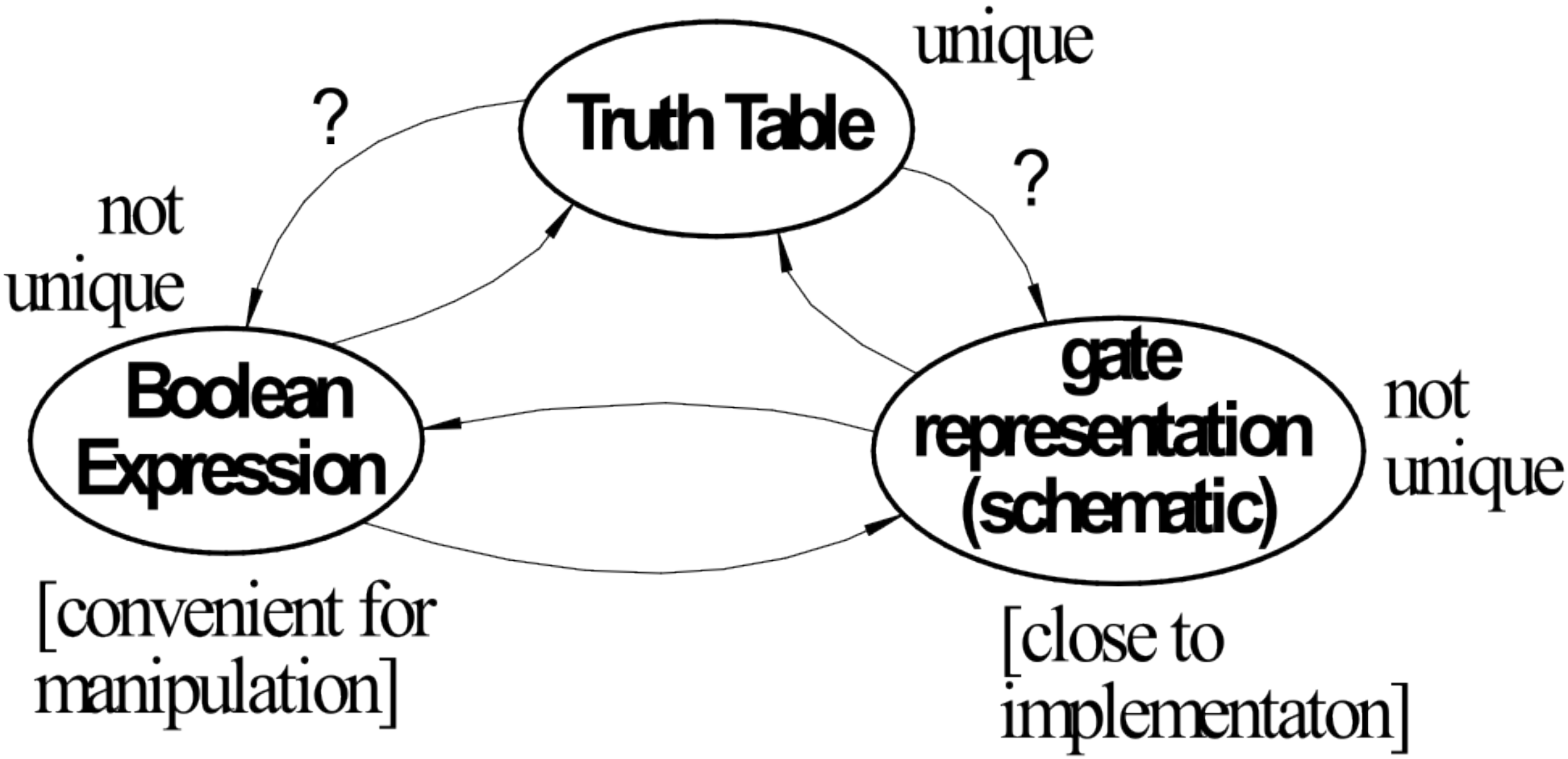
- Theorem: Any Boolean function that can be expressed as a truth table can be expressed using NAND and NOR.
  - Proof sketch:



  - How would you show that either NAND or NOR is sufficient?

Relationship Among Representations

* Theorem: Any Boolean function that can be expressed as a truth table can be written as an expression in Boolean Algebra using AND, OR, NOT.



Full-adder (FA) cell example

- Graphical Representation of FA-cell

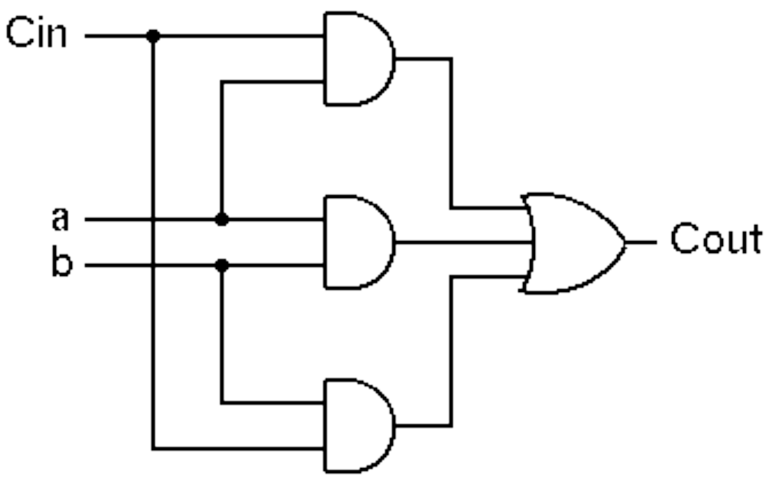
  $r_i = a_i \oplus b_i \oplus c_{in}$

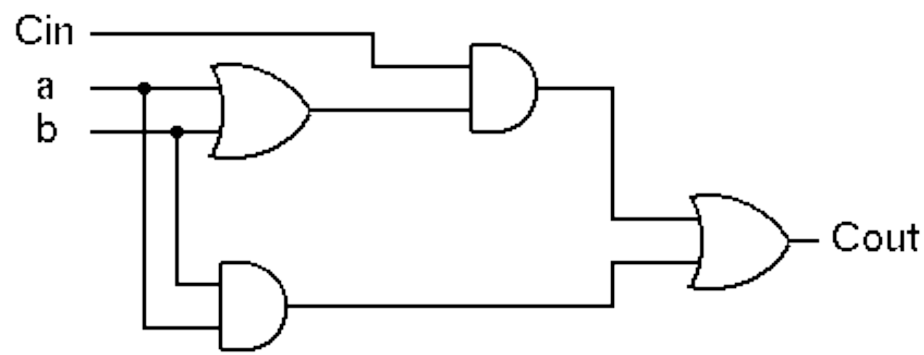
  $c_{out} = a_i c_{in} + a_i b_i + b_i c_{in}$

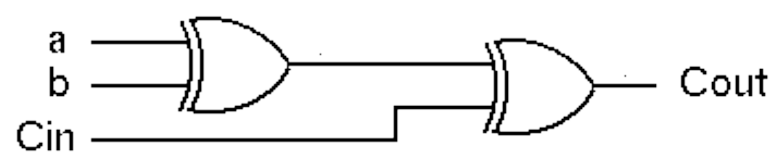
- Alternative Implementation (with only 2-input gates):

  $r_i = (a_i \oplus b_i) \oplus c_{in}$

  $c_{out} = c_{in}(a_i + b_i) + a_i b_i$







Boolean Algebra

Boolean Algebra

Set of elements B, binary operators $\{+, \bullet\}$, unary operation $\{\,'\,\}$, such that the following axioms hold:

1. B contains at least two elements a, b such that $a \neq b$.

2. Closure: a, b in B,

   $a + b$ in B, $a \bullet b$ in B, a' in B.

3. Communitive laws:

   $a + b = b + a, \quad a \bullet b = b \bullet a.$

4. Identities: 0, 1 in B

   $a + 0 = a, \quad a \bullet 1 = a.$

5. Distributive laws:

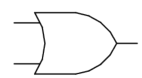
   $a + (b \bullet c) = (a + b) \bullet (a + c), \; a \bullet (b + c) = a \bullet b + a \bullet c.$

6. Complement:

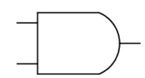
   $a + a' = 1, \; a \bullet a' = 0.$

$B = \{0,1\}$, $+$ = OR, $\bullet$ = AND, $'$ = NOT

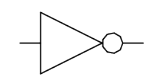
is a valid Boolean Algebra.



00	0
01	1
10	1
11	1



00	0
01	0
10	0
11	1



0	1
1	0

Some Laws (theorems) of Boolean Algebra

Duality: A dual of a Boolean expression is derived by interchanging OR and AND operations, and interchanging 0s and 1s (the variables are left unchanged).

$$\{F(x_1, x_2, \ldots, x_n, 0, 1, +, \bullet)\}^D = \{F(x_1, x_2, \ldots, x_n, 1, 0, \bullet, +)\}$$

Any law that is true for an expression is also true for its dual.

Operations with 0 and 1:

x + 0 = x   x * 1 = x

x + 1 = 1   x * 0 = 0

Idempotent Law:

x + x = x   x x = x

Involution Law:

(x')' = x

Laws of Complementarity:

x + x' = 1   x x' = 0

Commutative Law:

x + y = y + x   x y = y x

Some Laws (theorems) of Boolean Algebra (cont.)

Associative Laws:

$(x + y) + z = x + (y + z)$ $\qquad$ $x\,y\,z = x\,(y\,z)$

Distributive Laws:

$x\,(y + z) = (x\,y) + (x\,z)$ $\qquad$ $x + (y\,z) = (x + y)(x + z)$

"Simplification" Theorems:

$\boxed{x\,y + x\,y' = x}$ $\qquad$ $(x + y)\,(x + y') = x$

$x + x\,y = x$ $\qquad$ $x\,(x + y) = x$

$x + x'y = x + y$ $\qquad$ $x(x' + y) = xy$

DeMorgan's Law:

$(x + y + z + \ldots)' = x'y'z'$ $\qquad$ $(x\,y\,z\,\ldots)' = x' + y' + z'$

Theorem for Multiplying and Factoring:

$(x + y)\,(x' + z) = x\,z + x'\,y$

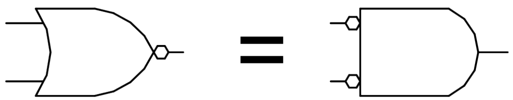
Consensus Theorem:

$x\,y + y\,z + x'\,z = (x + y)\,(y + z)\,(x' + z)$

$x\,y + x'\,z = (x + y)\,(x' + z)$

DeMorgan's Law

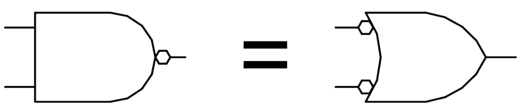
$$(x + y)' = x'\,y'$$



Exhaustive Proof

x	y	x'	y'	(x + y)'	x'y'
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

$$(x\,y)' = x' + y'$$

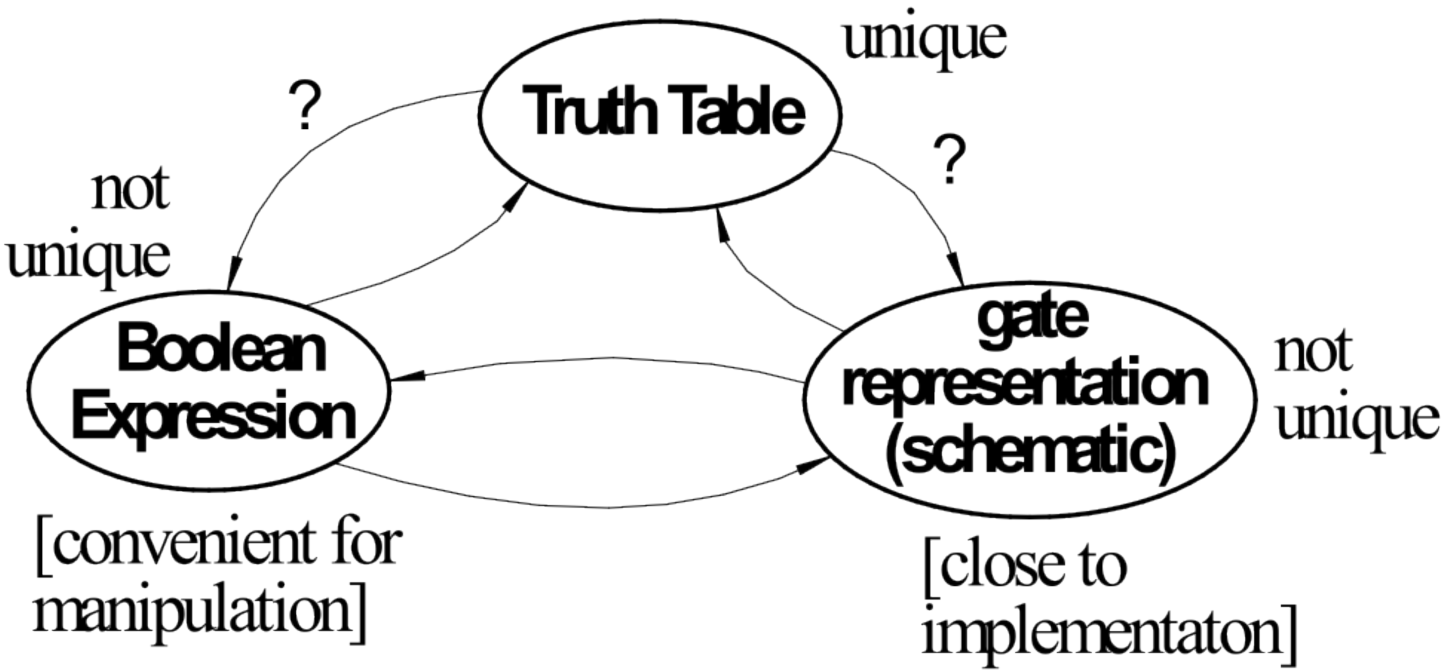


Exhaustive Proof

x	y	x'	y'	(x y)'	x' + y'
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

Relationship Among Representations

* Theorem: Any Boolean function that can be expressed as a truth table can be written as an expression in Boolean Algebra using AND, OR, NOT.



How do we convert from one to the other?

Canonical Forms

- Standard form for a Boolean expression - unique algebraic expression directly from a true table (TT) description.
- Two Types:
  - * **Sum of Products (SOP)**
  - * **Product of Sums (POS)**

- Sum of Products (disjunctive normal form, minterm expansion). Example:

Minterms	a	b	c	f	f'
a'b'c'	0	0	0	0	1
a'b'c	0	0	1	0	1
a'bc'	0	1	0	0	1
a'bc	0	1	1	1	0
ab'c'	1	0	0	1	0
ab'c	1	0	1	1	0
abc'	1	1	0	1	0
abc	1	1	1	1	0

One product (and) term for each 1 in f:

$f = a'bc + ab'c' + ab'c + abc' + abc$

$f' = a'b'c' + a'b'c + a'bc'$

(enumerate all the ways the function could evaluate to 1)

What is the cost?

Sum of Products (cont.)

Canonical Forms are usually not minimal:

Our Example:

```
f  = a'bc + ab'c' + ab'c + abc' +abc   (xy' + xy = x)
   = a'bc + ab' + ab
   = a'bc + a                          (x'y + x = y + x)
   = a + bc
```

```
f' = a'b'c' + a'b'c + a'bc'
   = a'b' + a'bc'
   = a' ( b' + bc' )
   = a' ( b' + c' )
   = a'b' + a'c'
```

Canonical Forms

- Product of Sums (conjunctive normal form, maxterm expansion).

Example:

maxterms	a	b	c	f	f'
a+b+c	0	0	0	0	1
a+b+c'	0	0	1	0	1
a+b'+c	0	1	0	0	1
a+b'+c'	0	1	1	1	0
a'+b+c	1	0	0	1	0
a'+b+c'	1	0	1	1	0
a'+b'+c	1	1	0	1	0
a'+b'+c'	1	1	1	1	0

*One sum (**or**) term for each **0** in f:*

f = (a+b+c)(a+b+c')(a+b'+c)

f' = (a+b'+c')(a'+b+c)(a'+b+c')
(a'+b'+c)(a+b+c')

(enumerate all the ways the function could evaluate to 0)

What is the cost?

Boolean Simplification

Algebraic Simplification Example

Ex: full adder (FA) carry out function (in canonical form):

Cout = a'bc + ab'c + abc' + abc

ci	a	b	r	co
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Algebraic Simplification

$$
\begin{aligned}
Cout &= a'bc + ab'c + abc' + abc \\
&= a'bc + ab'c + abc' + abc + abc \\
&= a'bc + abc + ab'c + abc' + abc \\
&= (a' + a)bc + ab'c + abc' + abc \\
&= (1)bc + ab'c + abc' + abc \\
&= bc + ab'c + abc' + abc + abc \\
&= bc + ab'c + abc + abc' + abc \\
&= bc + a(b' + b)c + abc' + abc \\
&= bc + a(1)c + abc' + abc \\
&= bc + ac + ab(c' + c) \\
&= bc + ac + ab(1) \\
&= bc + ac + ab
\end{aligned}
$$

Outline for remaining CL Topics

- K-map method of two-level logic simplification
- Multi-level Logic
- NAND/NOR networks
- EXOR revisited

Algorithmic Two-level Logic Simplification

Key tool: The Uniting Theorem:

$$xy' + xy = x(y' + y) = x(1) = x$$

ab	f
00	0
01	0
10	1
11	1

$f = ab' + ab = a(b' + b) = a$

b values change within the on-set rows

a values don't change

b is eliminated, a remains

ab	g
00	1
01	0
10	1
11	0

$g = a'b' + ab' = (a' + a)b' = b'$

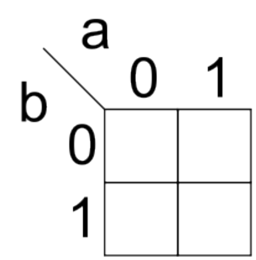
b values stay the same

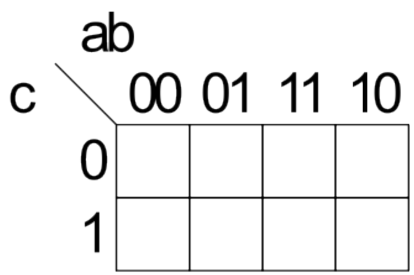
a values changes

b' remains, a is eliminated

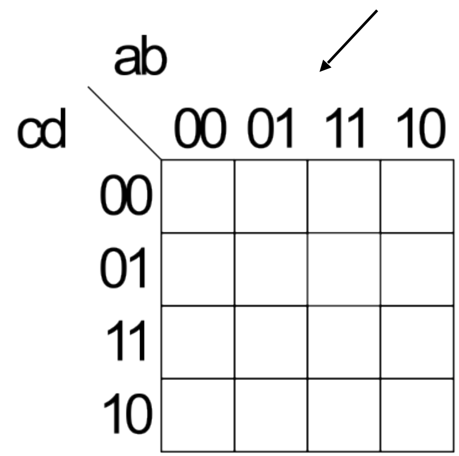
Karnaugh Map Method

- K-map is an alternative method of representing the TT and to help visual the adjacencies.





Note: "gray code" labeling.



5 & 6 variable k-maps possible

Karnaugh Map Method

- Adjacent groups of 1's represent product terms

b \ a	0	1
0	0	1
1	0	1

$f = a$

b \ a	0	1
0	1	1
1	0	0

$g = b'$

c \ ab	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$cout = ab + bc + ac$

c \ ab	00	01	11	10
0	0	0	1	1
1	0	0	1	1

$f = a$

K-map Simplification

1. Draw K-map of the appropriate number of variables (between 2 and 6)
2. Fill in map with function values from truth table.
3. Form groups of 1's.
   - ✓ Dimensions of groups must be even powers of two (1x1, 1x2, 1x4, …, 2x2, 2x4, …)
   - ✓ <u>Form as large as possible groups and as few groups as possible</u>.
   - ✓ Groups can overlap (this helps make larger groups)
   - ✓ Remember K-map is periodical in all dimensions (groups can cross over edges of map and continue on other side)
4. For each group write a product term.
   - the term includes the "constant" variables (use the uncomplemented variable for a constant 1 and complemented variable for constant 0)
5. Form Boolean expression as sum-of-products.

K-maps (cont.)

c \ ab	00	01	11	10
0	1	0	0	1
1	0	0	1	1

f = b'c' + ac

cd \ ab	00	01	11	10
00	1	0	0	1
01	0	1	0	0
11	1	1	1	1
10	1	1	1	1

f = c + a'bd + b'd'

(bigger groups are better)

Product-of-Sums K-map

1. Form groups of 0's instead of 1's.
2. For each group write a sum term.
   - the term includes the "constant" variables (use the uncomplemented variable for a constant 0 and complemented variable for constant 1)
3. Form Boolean expression as product-of-sums.

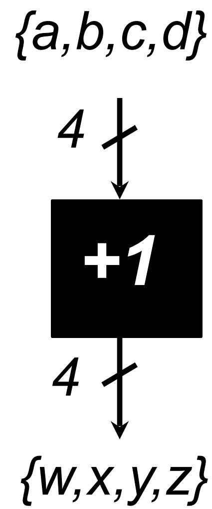
cd \ ab	00	01	11	10
00	1	0	0	1
01	0	1	0	0
11	1	1	1	1
10	1	1	1	1

$$f = (b' + c + d)(a' + c + d')(b + c + d')$$

BCD incrementer example

Binary Coded Decimal

	a b c d	w x y z
0	0 0 0 0	0 0 0 1
1	0 0 0 1	0 0 1 0
2	0 0 1 0	0 0 1 1
3	0 0 1 1	0 1 0 0
4	0 1 0 0	0 1 0 1
5	0 1 0 1	0 1 1 0
6	0 1 1 0	0 1 1 1
7	0 1 1 1	1 0 0 0
8	1 0 0 0	1 0 0 1
9	1 0 0 1	0 0 0 0
	1 0 1 0	- - - -
	1 0 1 1	- - - -
	1 1 0 0	- - - -
	1 1 0 1	- - - -
	1 1 1 0	- - - -
	1 1 1 1	- - - -



BCD Incrementer Example

- Note one map for each output variable.
- Function includes "don't cares" (shown as "-" in the table).
  - These correspond to places in the function where we don't care about its value, because we don't expect some particular input patterns.
  - We are free to assign either 0 or 1 to each don't care in the function, as a means to increase group sizes.
- In general, you might choose to write product-of-sums or sum-of-products according to which one leads to a simpler expression.

BCD incrementer example

w

cd \ ab	00	01	11	10
00	0	0	-	1
01	0	0	-	0
11	0	1	-	-
10	0	0	-	-

x

cd \ ab	00	01	11	10
00	0	1	-	0
01	0	1	-	0
11	1	0	-	-
10	0	1	-	-

y

cd \ ab	00	01	11	10
00	0	0	-	0
01	1	1	-	0
11	0	0	-	-
10	1	1	-	-

z

cd \ ab	00	01	11	10
00	1	1	-	1
01	0	0	-	0
11	0	0	-	-
10	1	1	-	-

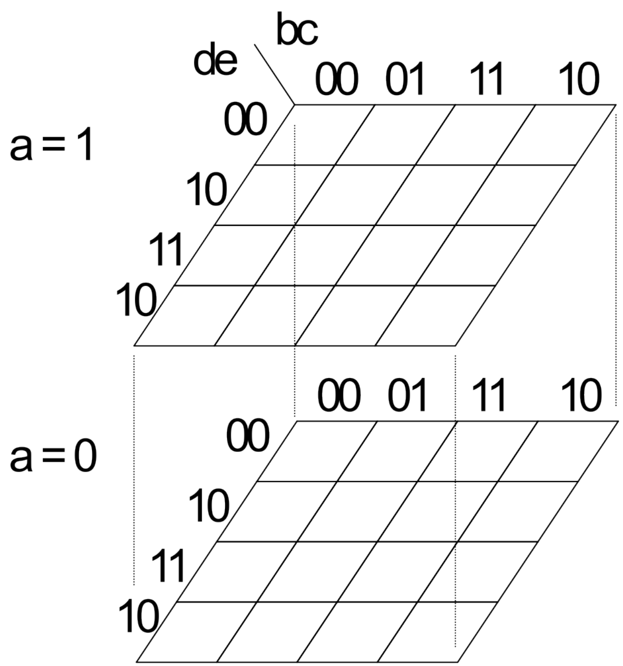
$w =$

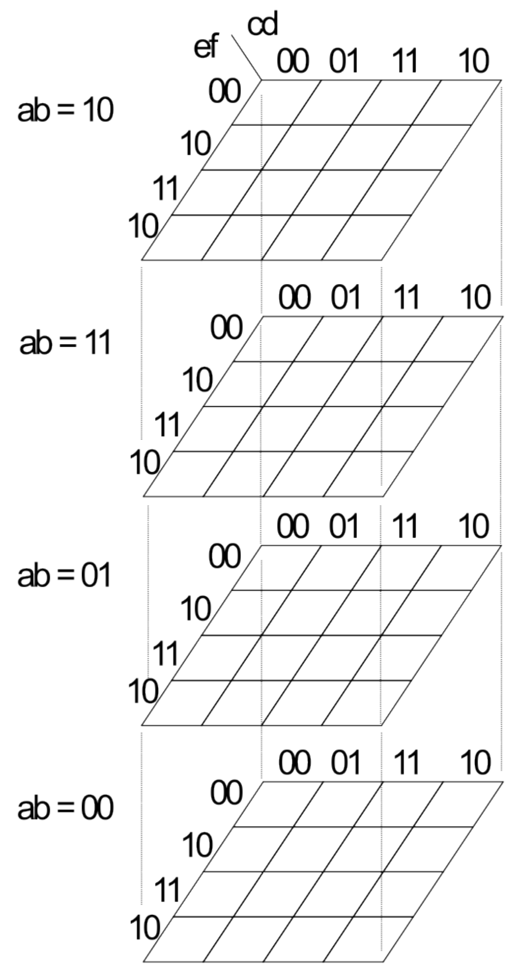
$x =$

$y =$

$z =$

Higher Dimensional K-maps





Boolean Simplification – Multi-level Logic

Multi-level Combinational Logic

- Example: reduced sum-of-products form

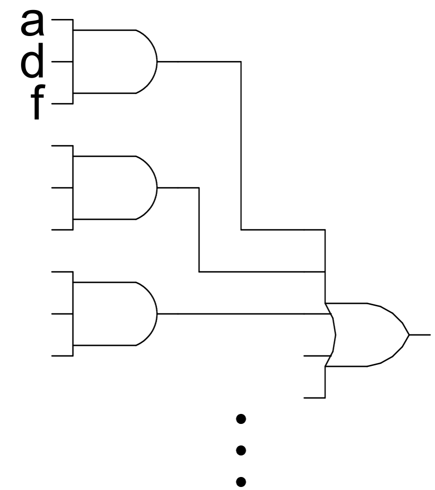
  `x = adf + aef + bdf + bef + cdf + cef + g`

- Implementation in 2-levels with gates:

  cost: 1 7-input OR, 6 3-input AND

  => ~50 transistors

  delay: 3-input OR gate delay + 7-input AND gate delay



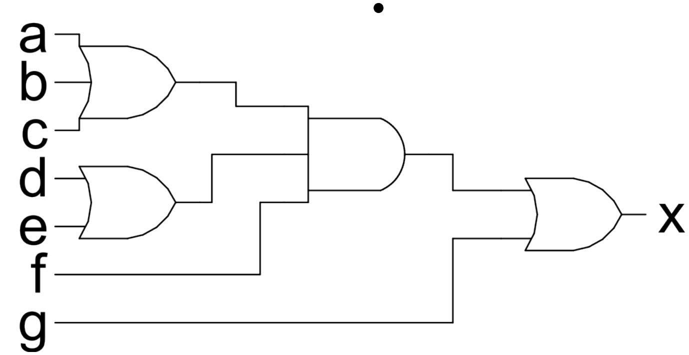
- Factored form:

  `x = (a + b +c)(d + e)f + g`

  cost: 1 3-input OR, 2 2-input OR, 1 3-input AND

  => ~20 transistors

  delay: 3-input OR + 3-input AND + 2-input OR



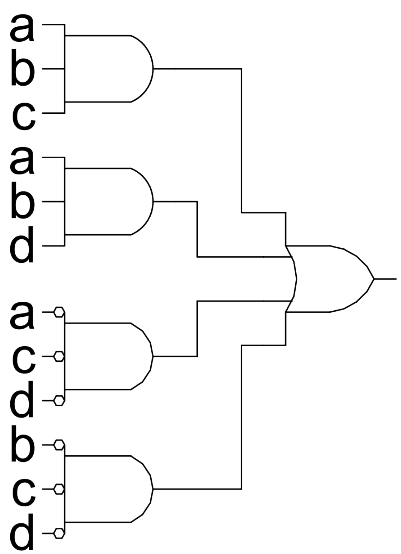
Footnote: NAND would be used in place of all ANDs and ORs.

Which is faster?

In general: Using multiple levels (more than 2) will reduce the cost. Sometimes also delay.
Sometimes a tradeoff between cost and delay.

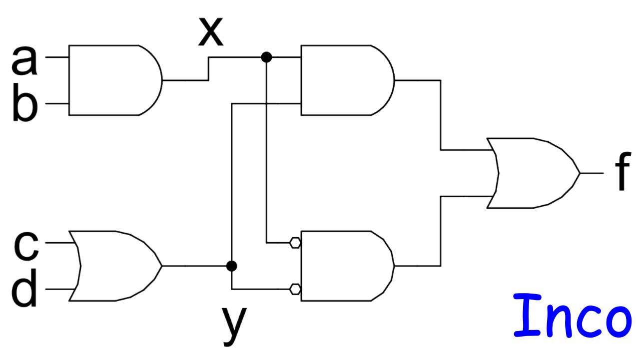
Multi-level Combinational Logic

Another Example: `F = abc + abd +a'c'd' + b'c'd'`



let `x = ab    y = c+d`

`f = xy + x'y'`



Incorporates fanout.

No convenient hand methods exist for multi-level logic simplification:

a) CAD Tools use sophisticated algorithms and heuristics

Guess what? These problems tend to be NP-complete

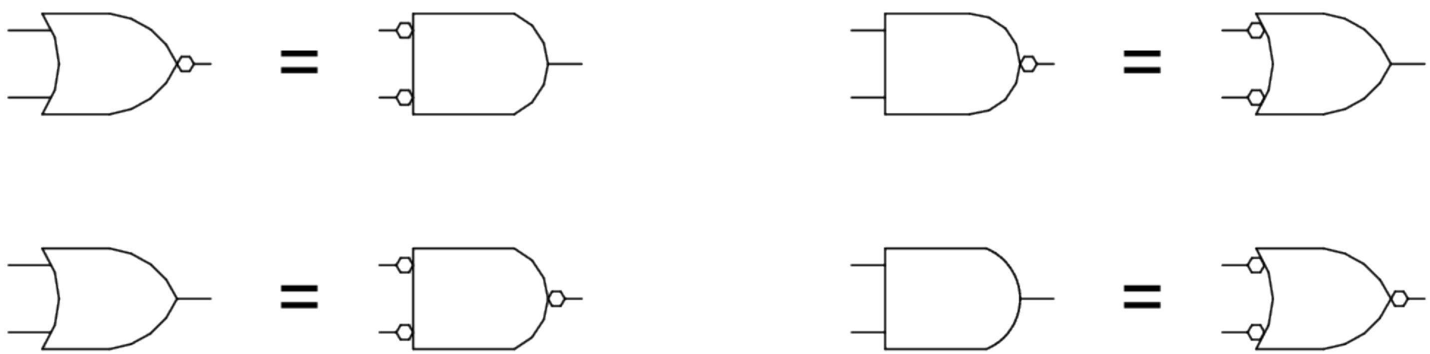
b) Humans and tools often exploit some special structure (example adder)

NAND-NAND & NOR-NOR Networks

DeMorgan's Law Review:

$(a + b)' = a'\, b'$ $\qquad$ $(a\, b)' = a' + b'$

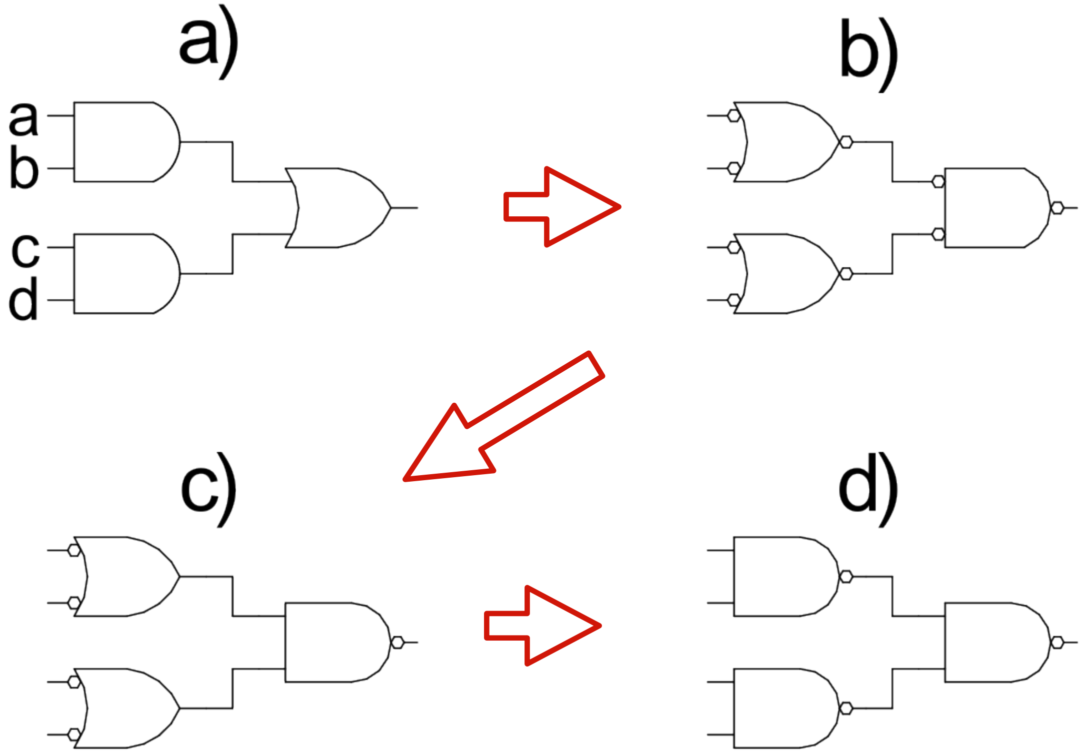
$a + b = (a'\, b')'$ $\qquad$ $(a\, b) = (a' + b')'$



push bubbles or *introduce in pairs* or *remove pairs:*
$(x')' = x$.

NAND-NAND & NOR-NOR Networks

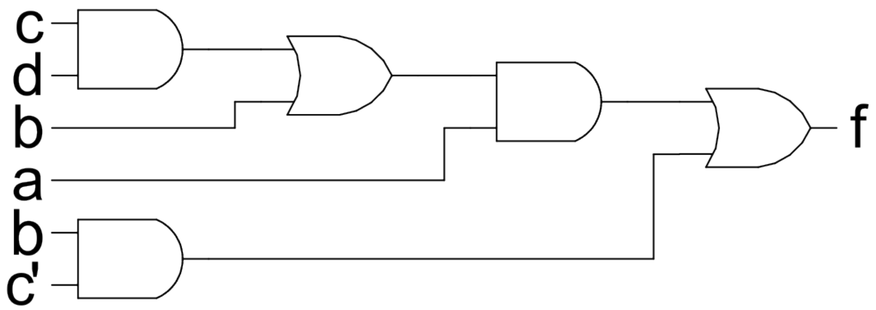
- Mapping from AND/OR to NAND/NAND



Multi-level Networks

Convert to NANDs:

$F = a(b + cd) + bc'$

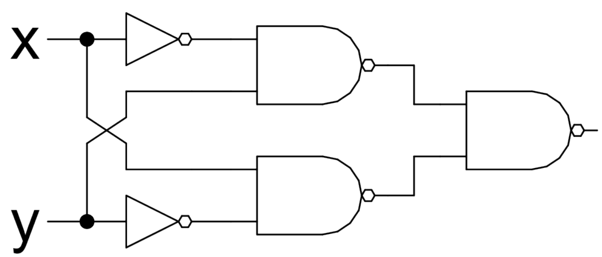


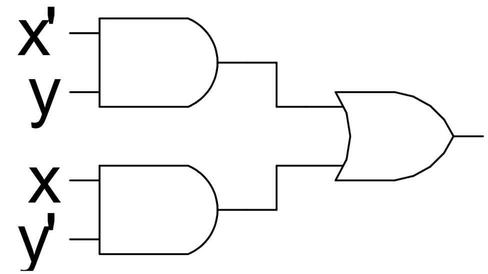
EXOR Function Implementations

Parity, addition mod 2

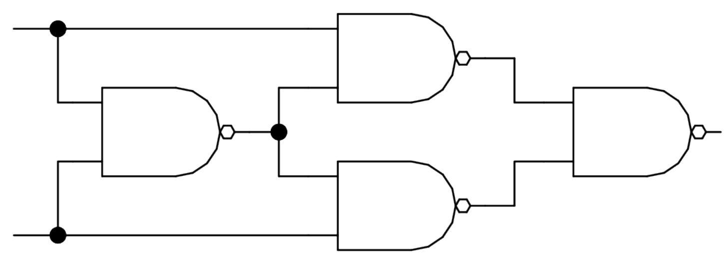
$x \oplus y = x'y + xy'$

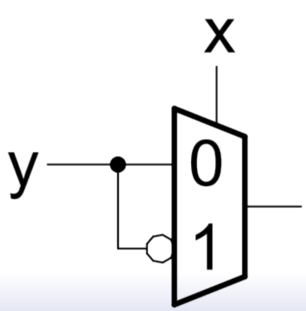
x y	xor	xnor
0 0	0	1
0 1	1	0
1 0	1	0
1 1	0	1





Another approach:





if x=0 then y else y'