EECS151/251A Discussion 12

Christopher Yarp

Apr. 26, 2019

Plan for Today

- Multipliers (including reminders from last lecture)
- Constant Multiplication
- Questions

Multipliers (From Last Week)

- Remember, the mechanics of multiplication in binary are generally the same as decimal multiplication (signed multiply requires a slight tweak).
- 2 Steps to Multiplication:
  - Generation of partial products
  - Adding partial products
- Making faster multipliers mostly involves changing how we deal with generating and adding the partial products

Unsigned Multiplication Example (From Last Week)

```
  4’b0011 (3)
* 4’b0110 (6)
```

- Partial Products can be generated in parallel

```
  4’b0011 (3)
* 4’b0110 (6)
  -------
     0000
    0011
   0011
+ 0000
---------
 00010010 (18)
```

Partial Products

Number Representations

- Unsigned Binary
  - Each bit place represents a different power of 2
  - Ex: 11 in unsigned binary = $2^3 + 2^1 + 2^0 = 8 + 2 + 1 = 11$

2^5	2^4	2^3	2^2	2^1	2^0
32	16	8	4	2	1
0	0	1	0	1	1

- Signed Binary – 2’s Complement
  - Each bit place still represents a different power of 2, except the most significant bit has negative weight
  - Converting to/from 2’s complement can be accomplished by performing a bitwise negation and adding 1. Ex. -11 in 2’s complement = $-2^5 + 2^4 + 2^2 + 2^0 = -32 + 16 + 4 + 1 = -11$

-2^5	2^4	2^3	2^2	2^1	2^0
-32	16	8	4	2	1
1	1	0	1	0	1

Signed Multiplication Example

```
  4’b0011  (3)
* 4’b1100 (-4)
```

- In 2’s Complement, the MSB is given negative weight
- Need to sign extend numbers when writing partial products
- Need to subtract partial product for MSB
- Carry bit of additions is discarded

```
   4’b0011  (3)
 * 4’b1100 (-4)
   -------
+ 00000000
+ 0000000
+ 000011
- 00011
 ---------
+ 00001100
+ 11100111+1
 ---------
+ 00001100
+ 11101000
 ---------
  11110100 (-12)
```

Signed Multiplication Example

```
  4’b1100 (-4)
* 4’b0011  (3)
```

- In 2’s Complement, the MSB is given negative weight
- Need to sign extend numbers when writing partial products
- Need to subtract partial product for MSB
- Carry bit of additions is discarded

```
   4’b1100 (-4)
 * 4’b0011  (3)
   -------
+ 11111100
+ 1111100
+ 000000
- 00000
 ---------
+ 11110100
+ 11111111+1
 ---------
+ 11110100
+ 00000000
 ---------
  11110100 (-12)
```

Signed Multiplication Example

```
  4’b1100 (-4)
* 4’b1101 (-3)
```

- In 2’s Complement, the MSB is given negative weight
- Need to sign extend numbers when writing partial products
- Need to subtract partial product for MSB
- Carry bit of additions is discarded

```
       4’b1100 (-4)
     * 4’b1101 (-3)
       --------
+   11111100
+   0000000
+   111100
-   11100
  ---------
+   11101100
+   00011111+1
  ---------
+   11101100
+   00100000
  ---------
(1)00001100 (12)
```

Signed Multiplication Example

```
  4’b0100 (4)
* 4’b0011 (3)
```

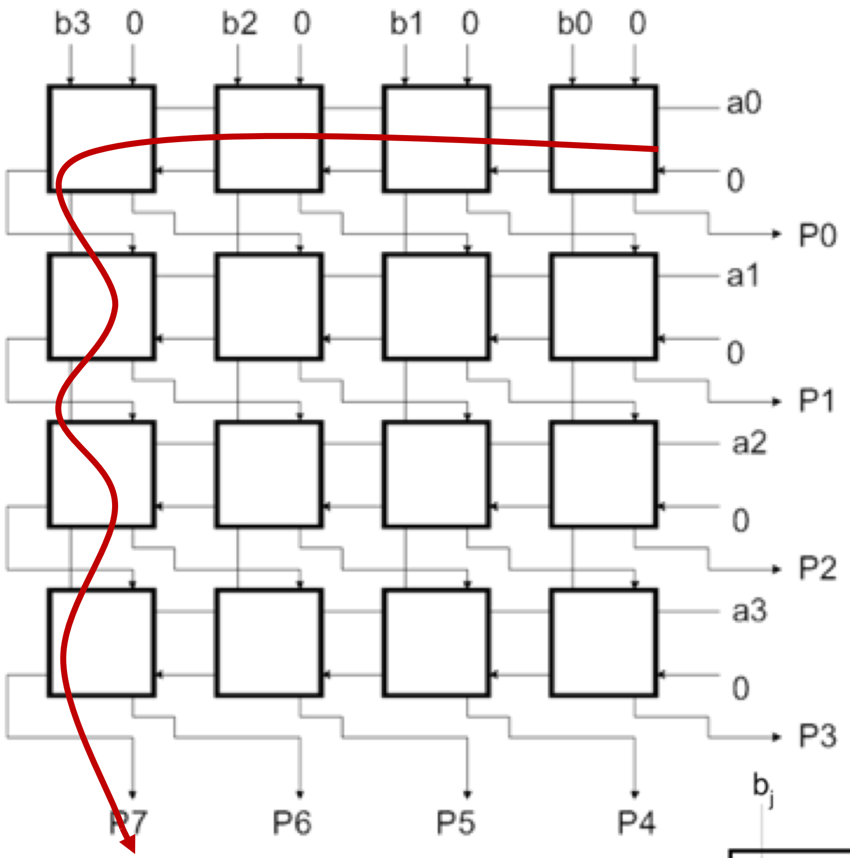
- In 2’s Complement, the MSB is given negative weight
- Need to sign extend numbers when writing partial products
- Need to subtract partial product for MSB
- Carry bit of additions is discarded

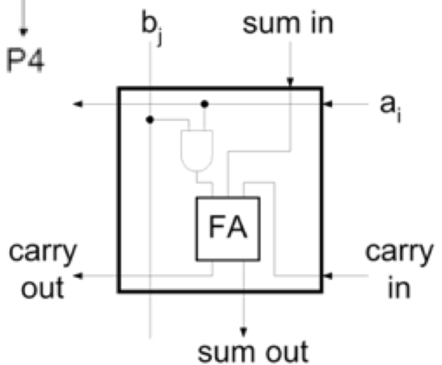
```
     4’b0100 (4)
   * 4’b0011 (3)
     -------
+  00000100
+  0000100
+  000000
-  00000
  ---------
+  00001100
+  11111111+1
  ---------
+  00001100
+  00000000
  ---------
   00001100 (12)
```

Accelerating Multiplication

Accelerating the Addition of Partial Products

- Let's look at an (unsigned) array multiplier
- The products can be computed in parallel but the carry chain when adding partial products is limiting the speed
- How do we improve performance without having a large increase in hardware?
  - We could implement each adder as a parallel prefix or a carry-lookahead adder
  - However, remember that these adders require more logic than a simple carry ripple adder

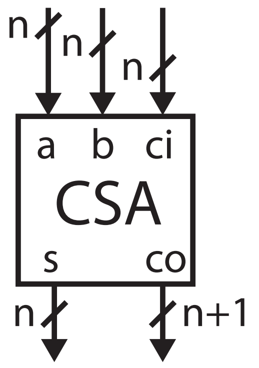


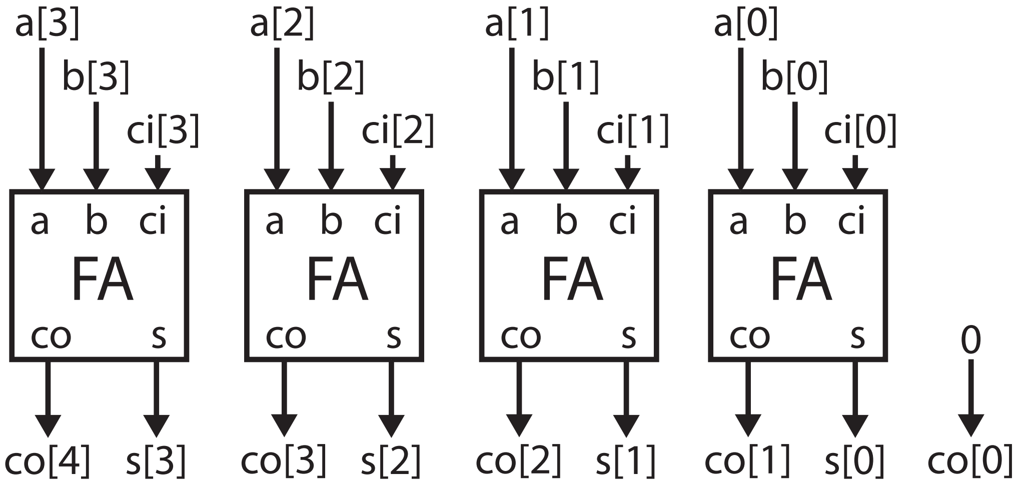


Figures from Lecture Slides

One Solution: Carry Save Addition

- When we generate a carry in a given column of an addition, we add it to the 2 values in the next column.
  - This addition may in turn generate its own carry
- If adding carries is just like another addition, can we delay adding the carry bits until later?
  - Yes, so long as we remember what the carry bits need to be added
- This is the basis of the carry save adder:
  - Takes in a, b, and carry_in (multi-bit)
  - Produces a sum and carry_out (multi-bit)





Using Carry Save Addition in Multipliers

- Carry now propagates down each column.
  - Carry ripple across rows is eliminated in the array
- Still need to handle carries at the end with a fast adder

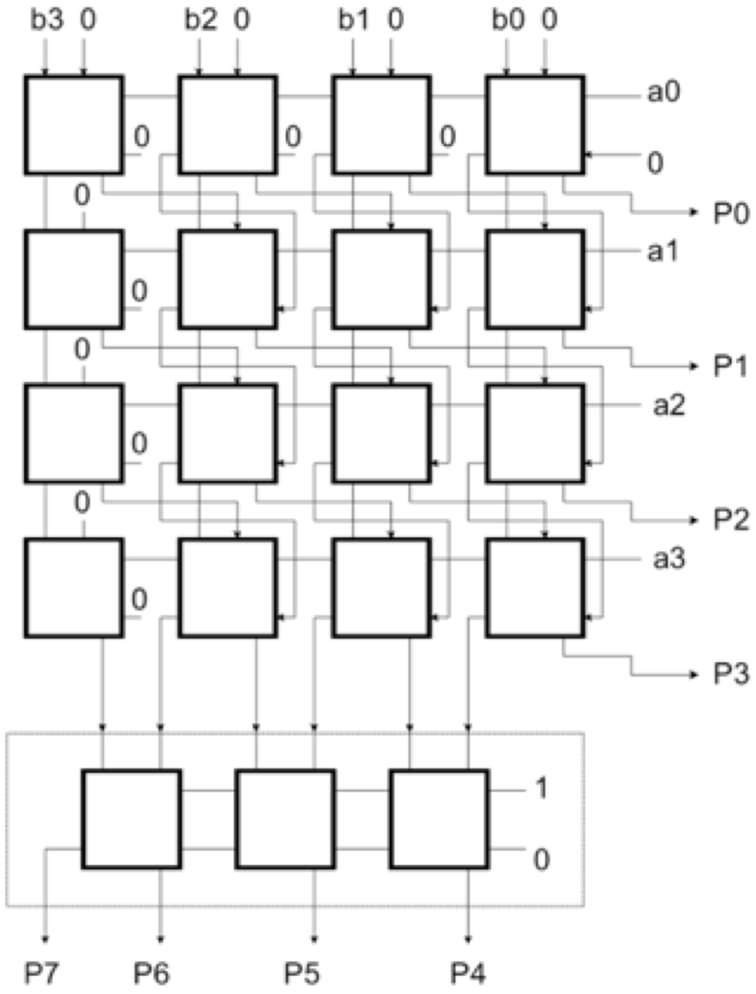
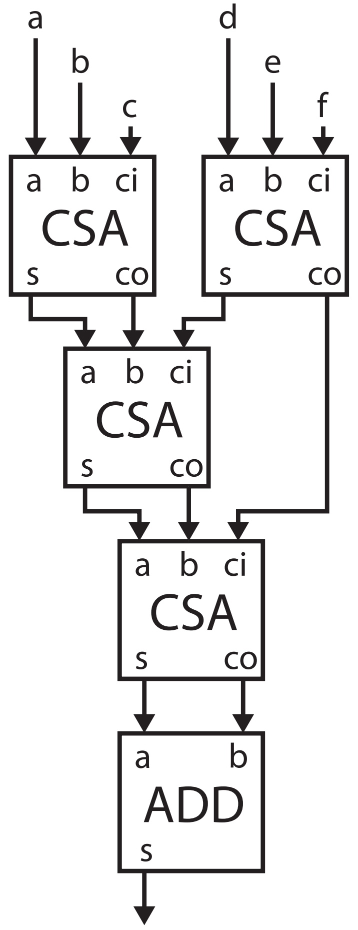


Figure from Lecture Slides

Using Carry Save Addition

- Remember, sums are associative and communitive.
- We can add the partial products in a tree structure using carry save adders!
  - Now have a number of layers that scales logarithmically!
- This is the basis of the Wallace Tree Multiplier



Radix and Multiplication

- Binary arithmetic has some advantages
  - Partial product generation is just a series of AND gates (including sign extension)
- However, there are also disadvantages
  - There is a partial product for each bit of the multiplier
  - That leads to a lot of partial products (a lot of additions)
- Ex. 3*4
  - single partial product in base 10
  - 4 partial products in base 2.
- Why don’t we consider a larger radix?

Radix 4 Multiplication

- Let’s consider 2 bits at a time
  - Halve the number of partial products we generate
- Radix 4 multiplication A*B
  - Partial Product Shift By 2 bits each time

B Digit	Partial Product	Partial Product (Rewritten)
0	0*A	0
1	1*A	A
2	2*A	4*A - 2*A
3	3*A	4*A - A

- Recall: Multiplications by powers of 2 are left shifts
- Can we use this property?

Booth Recoding

- Uses radix 4 arithmetic
- Modification: Partial Products for B==2 and B==3 can be separated into 4*A – {2, 1}A
- 4*A can be implemented as a shift to the left by 2
- 2*A can be implemented as a shift to the left by 1
- Recall that we are doing radix 4 multiplication, we shift left by 2 positions for the next partial product
- Therefore, any 4*A term can be handled in the next partial product!
    - To do this, the multiplier needs to look at 3 (rather than just 2) bits. The extra bit is the MSB of the previous

B Digit	Partial Product	Partial Product (Rewritten)
0	0*A	0
1	1*A	A
2	2*A	4*A - 2*A
3	3*A	4*A - A

Booth Recoding

B_{i+1}	B_i	B_{i-1}	Action	Comment
0	0	0	Add 0	
0	0	1	Add A	Includes +4*A from previous radix 4 digit = +A in this position due to left shift by 2
0	1	0	Add A	
0	1	1	Add 2*A	Includes +4*A from previous round (+A in this position). *2 is implemented as a left shift by 1
1	0	0	Sub 2*A	4*A will be added in when handling next radix 4 digit. *2 is implemented as a left shift by 1
1	0	1	Sub A	4*A will be added in when handling next radix 4 digit. Includes +4*A from previous radix 4 digit (+A in this position)
1	1	0	Sub A	4*A will be added in when handling next radix 4 digit.
1	1	1	Add 0	4*A will be added in when handling next radix 4 digit. Includes +4*A from previous radix 4 digit (+A in this position)

B Digit	Partial Product	Partial Product (Rewritten)
0	0*A	0
1	1*A	A
2	2*A	4*A - 2*A
3	3*A	4*A - A

Booth Recoding Example (Unsigned)

- Example: 6*4
- $B_{-1} = 0$

```
      4'b0110   (6)
   *  4'b0111   (7)
      --------
-         0110 ( Sub A)
+     01100    (Add 2A)
+    0000      ( Add 0)
    ----------
+    11111010 ( Sub A)
+     01100    (Add 2A)
+    0000      ( Add 0)
    ----------
(1)00101010 (42)
```

B_{i+1}	B_i	B_{i-1}	Action
0	0	0	Add 0
0	0	1	Add A
0	1	0	Add A
0	1	1	Add 2*A
1	0	0	Sub 2*A
1	0	1	Sub A
1	1	0	Sub A
1	1	1	Add 0

Additional Methods

- Pipelining!
  - Used in many high performance systems
  - Upside: Increased throughput
  - Downside: Increased latency
  - Good if you have many independent multiplications to perform and latency is acceptable

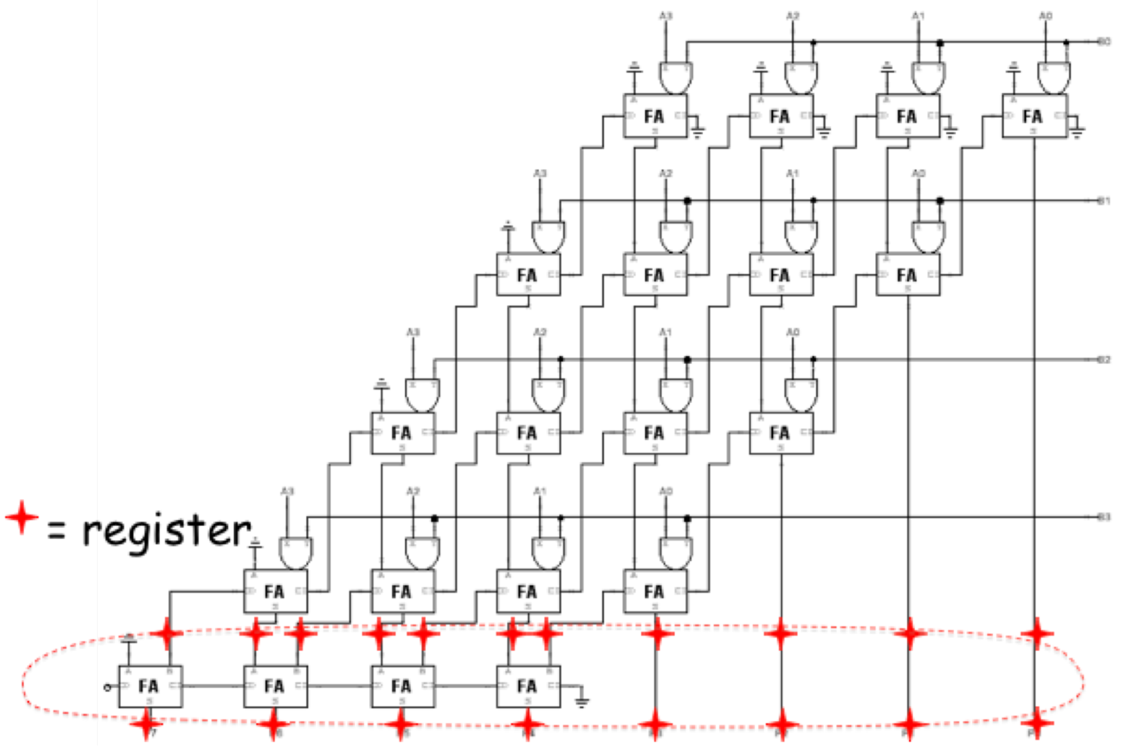


Figure from Lecture Slides

Signed Multiplication Tricks

- 2 things we need to do for signed multiplication:
  - Sign extend partial products
  - Subtract last partial products
- How can we simplify matters?
  - Sign extension requires additional logic
  - Add constants that allows us to eliminate the sign extension logic
  - Merge with the constant that is added when negating the last partial product

Trick with Sign Extension

- Ex. Sign Extend 1100 to 8 bits: 11111100
- Add 1000
- Causes a carry to ripple

```
    11111100
+   00001000
-----------
~~(1)~~00000100
```

- Results in the original input with the MSB Inverted

- Ex. Sign Extend 0100 to 8 bits: 00000100
- Add 1000
- No carry ripple

```
    00000100
+   00001000
-----------
    00001100
```

- Results in the original input with the MSB inverted
- Allows us to eliminate the 4 AND gates required for sign extension
- Need an inverter and to subtract the constant later

Application of Sign Extension Trick

1) Invert Last Partial Product (From Lecture Slides)

```
                                X3   X2   X1   X0
                              * Y3   Y2   Y1   Y0
                         --------------------
+ X3Y0 X3Y0 X3Y0 X3Y0 X3Y0 X2Y0 X1Y0 X0Y0
+ X3Y1 X3Y1 X3Y1 X3Y1 X2Y1 X1Y1 X0Y1
+ X3Y2 X3Y2 X3Y2 X2Y2 X1Y2 X0Y2
- X3Y3 X3Y3 X2Y3 X1Y3 X0Y3
-----------------------------------------
+ X3Y0 X3Y0 X3Y0 X3Y0 X3Y0 X2Y0 X1Y0 X0Y0
+ X3Y1 X3Y1 X3Y1 X3Y1 X2Y1 X1Y1 X0Y1
+ X3Y2 X3Y2 X3Y2 X2Y2 X1Y2 X0Y2
+ X3Y3 X3Y3 X2Y3 X1Y3 X0Y3    1    1    1
                                        1
-----------------------------------------
    Z7   Z6   Z5   Z4   Z3   Z2   Z1   Z0
```

2) Add Constants (From Lecture Slides)

```
                                X3   X2   X1   X0
                              * Y3   Y2   Y1   Y0
                         --------------------
+ X3Y0 X3Y0 X3Y0 X3Y0 X3Y0 X2Y0 X1Y0 X0Y0
+                         1    0    0    0
+ X3Y1 X3Y1 X3Y1 X3Y1 X2Y1 X1Y1 X0Y1
+                    1    0    0    0    0
+ X3Y2 X3Y2 X3Y2 X2Y2 X1Y2 X0Y2
+               1    0    0    0    0    0
+ X3Y3 X3Y3 X2Y3 X1Y3 X0Y3
+                         1 (+1 from Neg)
+          1    0    0    0    0    0    0
-          1    1    1    1    0    0    0
-----------------------------------------
    Z7   Z6   Z5   Z4   Z3   Z2   Z1   Z0
```

Application of Sign Extension Trick

3) Add Constants (From Lecture Slides)

$$
\begin{array}{ccccccccc}
 & & & & & X3 & X2 & X1 & X0 \\
 & & & & * & Y3 & Y2 & Y1 & Y0 \\
\hline
+ & & & & & \overline{X3Y0} & X2Y0 & X1Y0 & X0Y0 \\
+ & & & & \overline{X3Y1} & X2Y1 & X1Y1 & X0Y1 & \\
+ & & & \overline{X3Y2} & X2Y2 & X1Y2 & X0Y2 & & \\
+ & & X3Y3 & \overline{X2Y3} & \overline{X1Y3} & \overline{X0Y3} & & & \\
+ & & & & & 1 & \text{(+1 from Neg)} & & \\
- & & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hline
 & Z7 & Z6 & Z5 & Z4 & Z3 & Z2 & Z1 & Z0
\end{array}
$$

4) Negate Last Term (From Lecture Slides)

$$
\begin{array}{ccccccccc}
 & & & & & X3 & X2 & X1 & X0 \\
 & & & & * & Y3 & Y2 & Y1 & Y0 \\
\hline
+ & & & & & \overline{X3Y0} & X2Y0 & X1Y0 & X0Y0 \\
+ & & & & \overline{X3Y1} & X2Y1 & X1Y1 & X0Y1 & \\
+ & & & \overline{X3Y2} & X2Y2 & X1Y2 & X0Y2 & & \\
+ & & X3Y3 & \overline{X2Y3} & \overline{X1Y3} & \overline{X0Y3} & & & \\
+ & & & & & 1 & \text{(+1 from Neg)} & & \\
+ & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & Z7 & Z6 & Z5 & Z4 & Z3 & Z2 & Z1 & Z0
\end{array}
$$

Application of Sign Extension Trick

5) Add Constants (From Lecture Slides)

					X3	X2	X1	X0
				*	Y3	Y2	Y1	Y0
+					$\overline{X3Y0}$	X2Y0	X1Y0	X0Y0
+				$\overline{X3Y1}$	X2Y1	X1Y1	X0Y1	
+			$\overline{X3Y2}$	X2Y2	X1Y2	X0Y2		
+		X3Y3	$\overline{X2Y3}$	$\overline{X1Y3}$	$\overline{X0Y3}$			
+	1	0	0	1	0	0	0	0
	Z7	Z6	Z5	Z4	Z3	Z2	Z1	Z0

- Can be implemented with limited modifications to the unsigned multiplier!
- Requires passing some constants to full adders and inverting some terms

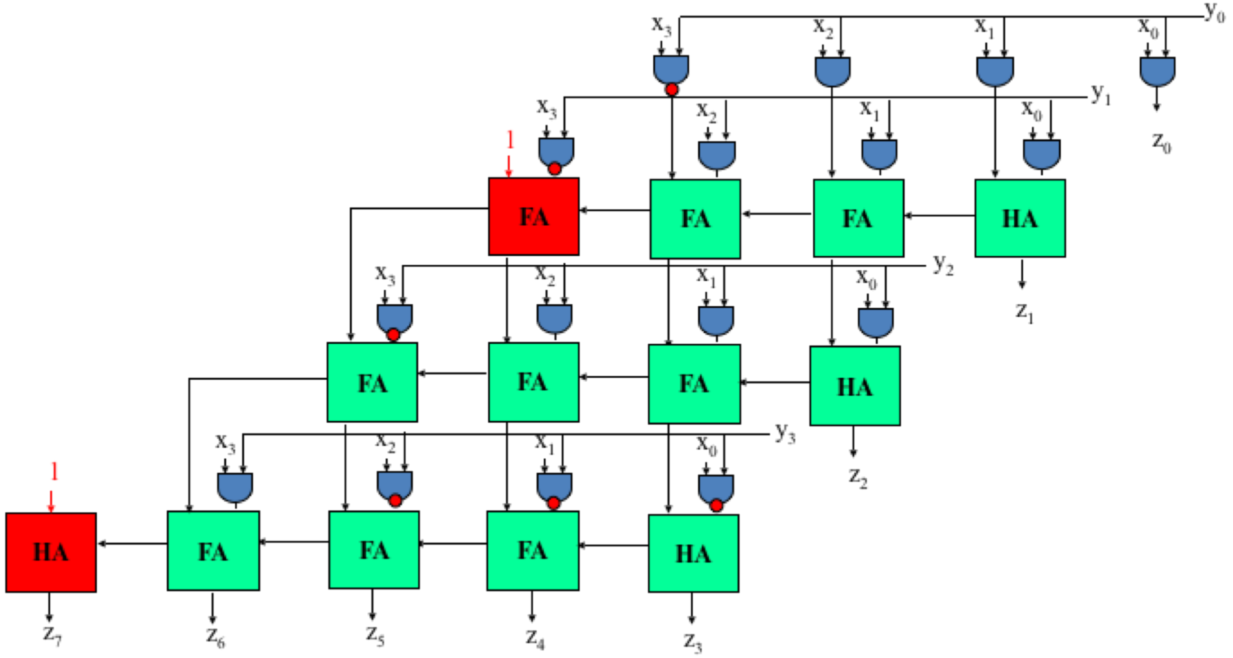


Figure from Lecture Slides

Constant Coefficient Multipliers

Multiplying by a Constant

- Observation: Every number can be factored into a sum of powers of 2
- This is exactly what we do when we write a number in binary!
  - Ex. $11 = 2^3 + 2^1 + 2^0 = 8 + 2 + 1$
- Can we leverage this to help us multiply by constants?
- Yes!

- Use the distributive property
  - Ex. $A*11 = A*(2^3 + 2^1 + 2^0) = A*2^3 + A*2^1 + A*2^0$
- Use the fact that power of 2 multiplies are shifts
  - Ex. A*11 = A<<3 + A<<1 + A<<0
  - Turned a multiply into shifts by fixed amounts and additions

Extending to Use Subtraction

- This concept can be extended to use subtraction
- Ex. $15 = 2^3 + 2^2 + 2^1 + 2^0 = 2^4 - 2^0 = 16 - 1$
- $A*15 = A*2^4 - A*2^0 = A<<4 - A<<0$
- This is denoted by drawing a line over digits with negative weight
- Ex. $15 = 001111 = 01000\overline{1}$

Canonical Signed Digit

- CSD Represents Numbers using 1, 0, $\bar{1}$ digits
- Minimizes the number of nonzero digits
  - Minimizes the number of additions needed when multiplying by a constant

Procedure (2 Passes):

1. Replace any occurrence of 2 or more 1's (01…10) with 10…$\bar{1}$0
2. Replace any occurrence of 2 or more 1's (01…10) with 10…$\bar{1}$0
   and Replace 01$\bar{1}$0 with 0010
   and Replace 0$\bar{1}$10 with 00$\bar{1}$0

Ex (From Lecture).

0010111 = 23

001100$\bar{1}$ (Pass 1)

010$\bar{1}$00$\bar{1}$ (Pass 2) = 32 – 8 – 1