Transients in First Order Circuits

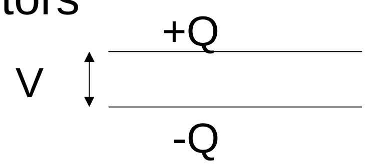
Lecture 7 review:

- Inductors and capacitors
- Energy storage

Today: (8.1)

- Step function input to RC first-order circuits
- R-L first-order circuits
- Close/open switch in first order circuits
- Rectangular pulse input to first order circuits

Capacitors



- Q = CV

  where Q=charge

  V=voltage difference between 2 plates

  C= capacitance

take derivative with respect to t on both sides

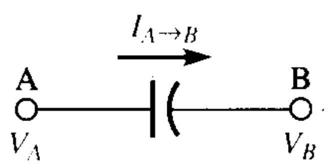
- dQ/dt = C dV/dt = i
- i=CdV/dt (remember I-V diagram)
- current = constant X time derivative of voltage
- Ohm's law tells us about the relationship between

V and I for a resistor. This equation describes the relationship between i and V for a capacitor.

Capacitors (continue)

- The I-V relationship for a capacitor is:

$$I_{A\to B} = C\frac{d}{dt}(V_{AB})$$



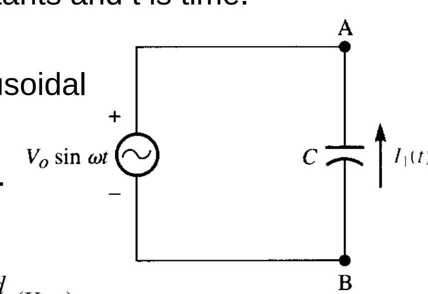
Where C is the capacitance in Farad or F, mF, μF, nF, pF

Notice the current depends on the derivative. If the derivative is zero, then there is no current. The derivative is zero when the voltage remains constant and does not change with time. An example would be: dc circuit.

∴No current goes through a capacitor in a dc circuit.

Capacitor example

- Find the current $I_1(t)$ that passes through the capacitor as shown. The voltage source is a sinusoid $V_0 \sin\omega t$, where V_0 and ω are given constants and t is time.



Since the voltage source is sinusoidal (change with time), the current across the capacitor is nonzero.

From circuit $V_A - V_B = V_0 \sin\omega t$

From previous page $I_{A\to B} = C\frac{d}{dt}(V_{AB})$

$$I_{A\to B} = C\frac{d}{dt}(V_0 \sin\omega t)$$

$$I_{A\to B} = CV_0\omega\cos\omega t$$

But i is from B to A direction:

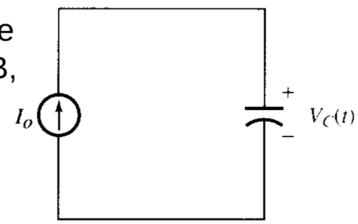
$$I_{B\to A} = I_1 = -CV_0\omega\cos\omega t$$

Capacitor example

- Find the $v_c(t)$ across the capacitor as shown. The current I_0 through the current source is constant.

Apply the I-V equation for capacitor from the previous page (when the current direction A->B, then Voltage is V_{AB})

$$I_{A\to B} = C\frac{d}{dt}(V_{AB})$$

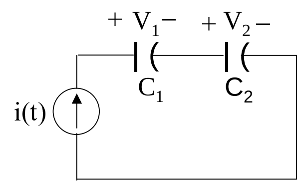


In the circuit on the right, current I_0 is entering cap.

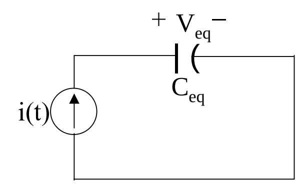
$$I_0 = C\frac{d}{dt}(V_c) \Rightarrow \frac{I_0}{C}dt = dV_c \Rightarrow V_c = \int dV_c = \frac{I_0}{C}\int dt = \frac{I_0 t}{C} + K$$

$\therefore$ The voltage is increasing proportional to time, cap. Is charged.

CAPACITORS IN SERIES



Equivalent to



Equivalent capacitance defined by

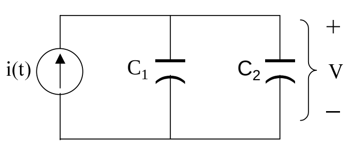
$$V_{eq} = V_1 + V_2 \text{ and } i = C_{eq}\frac{dV_{eq}}{dt} = C_{eq}\frac{d(V_1+V_2)}{dt}$$

$$i = C_1\frac{dV_1}{dt} = C_2\frac{dV_2}{dt}$$

$$\text{So } \frac{dV_1}{dt} = \frac{i}{C_1}, \quad \frac{dV_2}{dt} = \frac{i}{C_2}, \qquad \text{so } \frac{dV_{eq}}{dt} = i\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \equiv \frac{i}{C_{eq}}$$

Clearly, $\boxed{C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}} = \dfrac{C_1 C_2}{C_1 + C_2} \qquad \text{CAPACITORS IN SERIES}}$

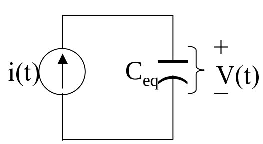
CAPACITORS IN PARALLEL



$$i(t) = C_1\frac{dV}{dt} + C_2\frac{dV}{dt}$$

Equivalent capacitance defined by

$$i = C_{eq}\frac{dV}{dt}$$

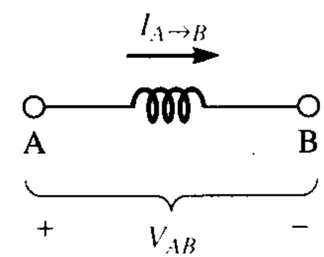


Clearly, $\boxed{C_{eq} = C_1 + C_2 \qquad \text{CAPACITORS IN PARALLEL}}$

Inductors

- Ideal inductor is a 2-terminal device.

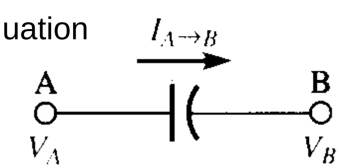
$$V_{AB} = L\frac{d}{dt}I_{A\to B}$$



- Where L is a constant called inductance with unit in Henry or H, mH, µH, nH.

Notice similarity with capacitance equation

$$I_{A\to B} = C\frac{d}{dt}(V_{AB})$$



Inductors (example)

- Assume there is no current going thru an inductor, at time =0, a time varying current i(t) is applied thru the inductor terminals. What is the voltage across the inductor terminals as a function of time.

Apply the inductor equation in the previous page:

$$v(t) = L\frac{d}{dt}i(t)$$

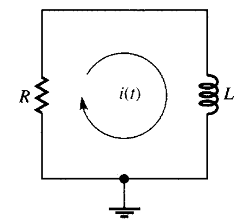
If a time varying voltage v(t) is applied across its terminals. What is the current thru the inductor as a function of time.

Again, apply the inductor equation in the previous page:

$$v(t) = L\frac{d}{dt}i(t) \Rightarrow di(t) = \frac{1}{L}v(t)dt \Rightarrow i(t) = \int di(t) = \int_0^t \frac{1}{L}v(t)dt$$

Inductance (example)

- Write a loop equations for the loop current i(t)



The voltage drop in the inductor is: $L\frac{d}{dt}i(t)$

The voltage drop in the resistor is: $i(t)R$

So, the loop equation (KVL) is: $L\frac{d}{dt}i(t) + i(t)R = 0$

What if you have the same circuit in series with a capacitor C?

Recall the I-V eq. for Capacitor: $i(t) = C\frac{d}{dt}v(t)$

$$i(t) = C\frac{d}{dt}v(t) \Rightarrow dv(t) = \frac{1}{C}i(t)dt \Rightarrow v(t) = \int \frac{1}{C}i(t)dt$$

The voltage drop in the capacitor is: $v(t) = \int \frac{1}{C}i(t)dt$

So, the new loop equation (KVL) is: $L\frac{d}{dt}i(t) + i(t)R + \int \frac{1}{C}i(t)dt = 0$

Parallel/Series Inductors

- Inductor in series is similar to resistor (sum):

$$L_{series} = L_1 + L_2$$

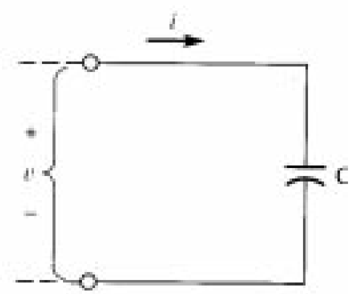
- Inductor in parallel is similar to resistor (product over sum):

$$L_{parallel} = \frac{L_1 L_2}{L_1 + L_2}$$

Energy Storage

	Element		equation	energy+ or -?	energy eq.
•	Resistor	R	V=IR	dissipate energy	V^2/R or I^2R
•	Capacitor	C	I=CdV/dt	stored	???
•	Inductor	L	V=LdI/dt	stored	???

Assume the capacitor is uncharged, at t=0, a voltage v(t) is applied. The instantaneous power enter the capacitor is: p(t)=v(t)i(t)



The energy enter the capacitor (from time=0 to t)is:

$$E = \int_0^t p(t)dt = \int_0^t v(t)i(t)dt$$

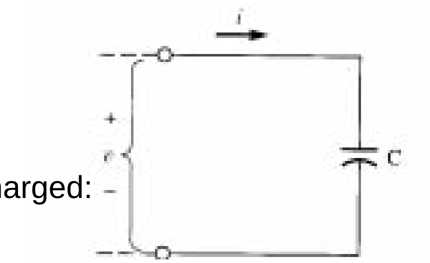
Energy Storage (continue)

$$E=\int_0^t p(t)dt=\int_0^t v(t)i(t)dt$$

Recall the I-V eq. for Capacitor: $i(t)=C\frac{d}{dt}v(t)$

$$E=\int_0^t vidt=\int_0^t vC\frac{dv}{dt}dt=\int_0^t \frac{1}{2}Cd(v^2)$$

$$E=\frac{1}{2}C[v(t)]^2-\frac{1}{2}C[v(0)]^2$$



If the Capacitor is initially uncharged:

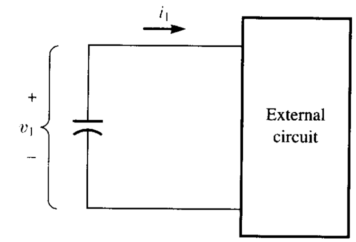
$$E=\frac{1}{2}C[v(t)]^2$$

$\therefore$ Energy store in a capacitor is: 1/2 CV^2

Energy Storage (continue)

- Once energy is stored in capacitor, is there way we can regain the energy?

Consider the circuit on the right. Suppose the capacitor is initially charged to voltage V, is to discharge to an external circuit.



The energy recovered from the capacitor (entered the external circuit) after an infinite length of time:

$$E=\int_0^\infty p(t)dt=\int_0^\infty v_1(t)i_1(t)dt$$

$$E=\int_0^\infty v(t)\left[-C\frac{dv}{dt}\right]dt=-C\int_0^\infty v(t)dv=-C\int_0^\infty d\left(\frac{v^2}{2}\right)$$

Minus sign is due to the reference direction

$$E=-\frac{1}{2}C\left[v(\infty)^2-v(0)^2\right]$$

The voltage is fully discharged (V=0) when t=∞

$$E=\frac{1}{2}C\left[v(0)^2\right]$$

Energy storage for inductors

$$E=\int_0^t v(t)i(t)dt=\int_0^t L\frac{di}{dt}idt=\int_0^t \frac{1}{2}Ld(i^2)$$

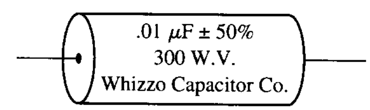
$$E=\frac{1}{2}L[i(t)]^2-\frac{1}{2}L[i(0)]^2$$

$$E=\frac{1}{2}LI^2$$

Where I is the final current at time t, assumed the current through the inductor is zero.

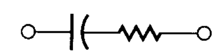
Practical Capacitors and inductors

Practical capacitor = ideal capacitor in series with a resistor



(a)

The resistor part dissipates energy, thus, practical capacitor can never retain energy definitely, e.g. every DRAM cell need to be refreshed periodically to retain its value.



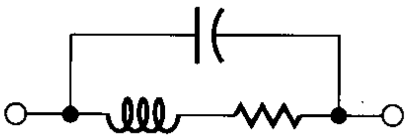
(b)

Capacitors use below 1GHz: mica, ceramic, and tantalum (see Figure a). Capacitors are specified by their capacitance value, maximum voltage applied across terminals, their tolerance.

Practical Capacitors and inductors (cont)

- a practical inductor can be replaced by an ideal inductor in series with a resistor and then in parallel with a capacitor

Again, practical inductor can dissipate energy because of the present of the resistor.



Transients in First Order Circuits

- Definition
  – transient (transitional, change…)
  – steady state
- First order circuits?
  – Circuits which can be characterized by first order differential equations, e.g. RC ckt, RL ckt…
  – What about RLC ckt?

Several Rules with Circuits

- Rule 1
  – The voltage across a capacitor cannot change instantaneously
  – i = C dv/dt
- Rule 2
  – The current through an inductor cannot change instantaneously
  – v = L di/dt
- Rule 3
  – In the dc steady state the current through a capacitor is zero.
- Rule 4
  – In the dc steady state the voltage across an inductor is zero.

<u>In summary:</u>

	C	L
Quantity that cannot be discontinuous	voltage	current
Quantity that is zero in the dc steady state	current	voltage

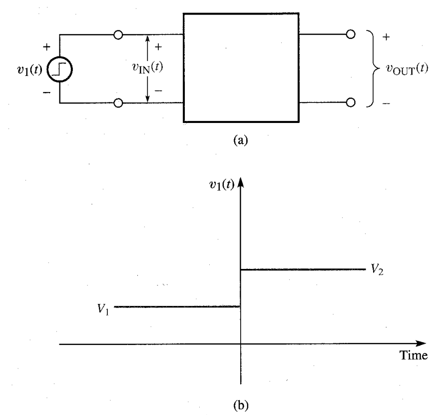
Transient of first order circuits (FOC)

- Step response of R-L or R-C circuits?
- General form of the response:

$$V_{out}(t) = A + Be^{-t/\tau}$$

for t>0

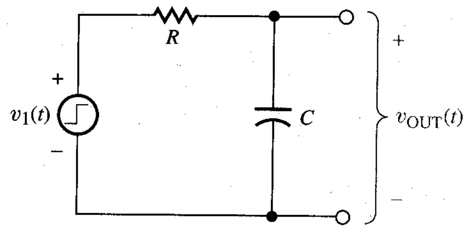
where A and B are some constants, τ is the time constant depends on the value of R-L or R-C.



RC circuits

- Find the transient (step) response of the RC circuit as shown

$v_1(t) = 0$ for $t<0$
$= V$ for $t>0$



Write nodal equation (KCL) at the + terminal:

$$\frac{v_1(t) - v_{out}(t)}{R} - C\frac{dv_{out}}{dt} = 0$$

Which can be rewritten as: $$\frac{dv_{out}}{dt} + \frac{1}{RC}v_{out}(t) = \frac{1}{RC}v_1(t) \quad (1)$$

Recall the solution will have the form: $$v_{out}(t) = A + Be^{-\frac{t}{\tau}} \quad (2)$$

Can substitute (2) into (1) and determine A and B and τ

RC Circuits (continue)

Substitute $v_{out}(t) = A + Be^{-\frac{t}{\tau}}$

into $$\frac{dv_{out}}{dt} + \frac{1}{RC}v_{out}(t) = \frac{1}{RC}v_1(t)$$

becomes $$-\frac{B}{\tau}e^{-t/\tau} + \frac{A}{RC} + \frac{B}{RC}e^{-t/\tau} = \frac{1}{RC}V$$ (let $v_1(t)=V$, for $t>0$)

Can be rewritten as: $$\left(\frac{A}{RC} - \frac{V}{RC}\right) + \left(\frac{B}{RC} - \frac{B}{\tau}\right)e^{-t/\tau} = 0$$

Which can be satisfied if: $\left(\frac{A}{RC} - \frac{V}{RC}\right) = 0$ and $\left(\frac{B}{RC} - \frac{B}{\tau}\right)e^{-t/\tau} = 0$

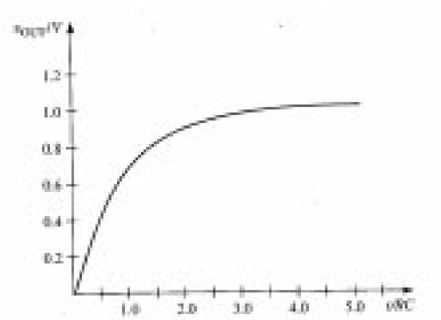
$A = V$ $\quad \tau = RC$ $\quad$ B=? $\quad$ Use $v_{out}(t) = A + Be^{-\frac{t}{\tau}}$

Recall rule 1(voltage can not change instantaneously):

$$v_{out}(0+) = A + Be^{-\frac{0}{\tau}} = A + B = v_{out}(0-) = 0 \quad \Rightarrow B = -A$$

RC Circuits (continue)

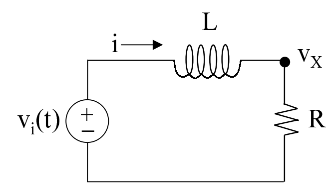
$$v_{out}(t) = V - Ve^{-\frac{t}{RC}}$$
$$v_{out}(t) = V(1 - e^{-\frac{t}{RC}})$$

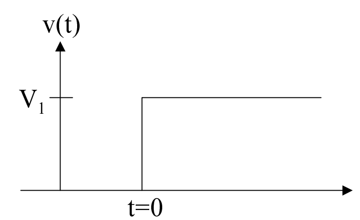


Notice the second term starts from V and exponentially decay to 0. It will never reach 0, but approach to 0 asymptotically.

As a result, v_{out} starts from 0 and asymptotically approach to V.

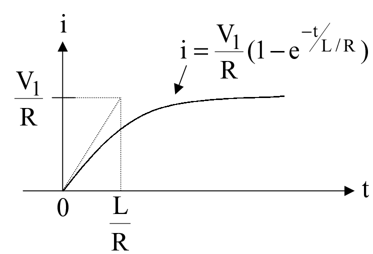
THE BASIC INDUCTOR CIRCUIT

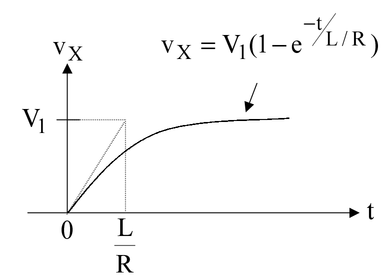




KVL: $v_i = L_i\frac{di}{dt} + iR$ $\qquad$ But $v_X = iR$

$$v_i = \frac{L}{R}\frac{dv_X}{dt} + v_X \qquad \text{SOLN for } t>0 \text{ given } i(0) = 0$$





RC Circuits (continue)

Steps to find the step-input response of a first order circuit:

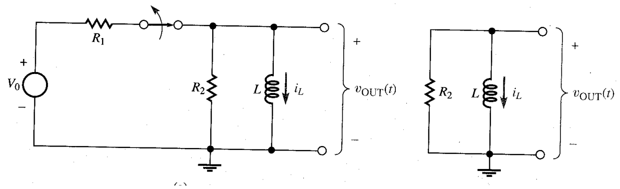
- Write a node equation or loop equation
- Substitute the general solution $v_{out}(t) = A + Be^{-\frac{t}{\tau}}$

into the node/loop equation to obtain the A/B/τ unknowns.

- Use the initial condition of circuit (and rule 1/2) to obtain the third equation.
- Thus, you have 3 equations and 3 unknowns

Open/Close switch in FOC

- Typically these switches are not mechanical switch as shown, but electronic switch (e.g. transistor).



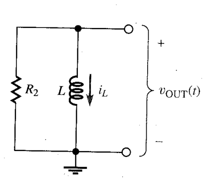
In the circuit above, the switch is closed for all time t<0 (left) and open t > 0 (right circuit). What is $v_{out}(t)$ at t < 0 and t > 0?

t<0, (steady state)from rule 4, $v_{out}(t) = 0 \Rightarrow i_L = V_0/R_1$ (why?)

From rule 2 (current can not change inst.): $i_L(0-) = i_L(0+) = V_0/R_1$

When t>0, the voltage source is dropped (see ckt in the right)

Open/Close switch in FOC (continue)



- When t>0, the voltage source is dropped
- From L circuit equation: $v_{out}(t) = L\frac{di_L}{dt}$
- From KVL: $\frac{v_{out}}{R_2} = -i_L$ (@)

(-ve because current thru the resistor is opposite of i_L.)
substitute the second eq. to the first.

$$\frac{dv_{out}}{dt} + \frac{v_{out}}{L/R_2} = 0$$

substitute the general first order solution $v_{out}(t) = A + Be^{-\frac{t}{\tau}}$ into this eq.

$$v_{out}(t) = Be^{-t/(L/R_2)}$$

$i_L = V_0/R_1$ (from the previous page)

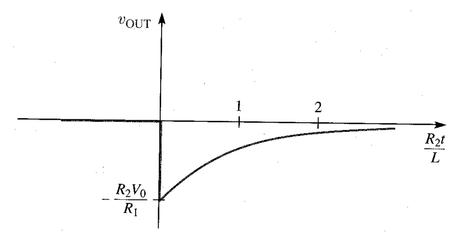
B can be found by setting $v_{out}(0+) = -i_L R_2$ (from eq. @) $= -V_0 R_2/R_1$

Open/Close switch in FOC (continue)

$v_{out}(t) = Be^{-t/(L/R_2)}$ and $i_L = V_0/R_1$ and $v_{out}(0+) = -i_L R_2$

$$\Rightarrow \quad v_{out}(t) = -\frac{R_2 V_0}{R_1} e^{-t/(L/R_2)}$$

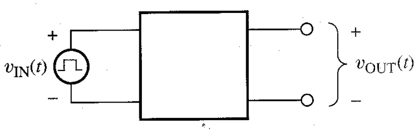
The plot on the right show the output V_{out}.

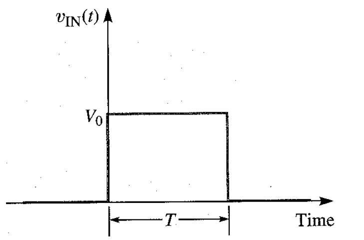


$\therefore$Notice, it is possible to raise R_2 to be very large, thus, v_{out} could be very large proportionally. In automobile spark plug, 12v from the battery can be raised to thousand of volt using this technique.

Response to a rectangular pulse

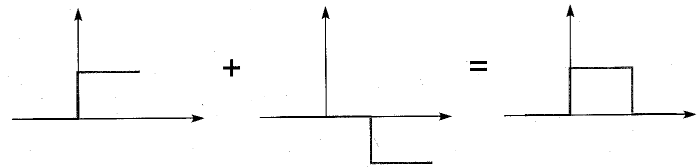
We learn to analyze first order circuit response to step input. What about rectangular pulse? (digital signal is more like rectangular pulse than step function!)

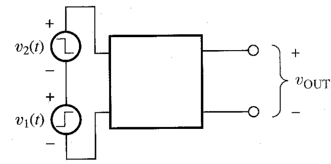


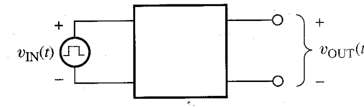


Response to a rectangular pulse (continue)

It is possible to break the rectangular pulse into 2 step functions as below:



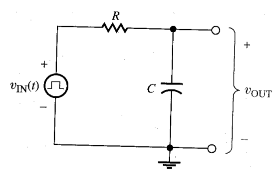




The rectangular function = step function (v_1) + delayed step function (v_2) with a negative coeff.

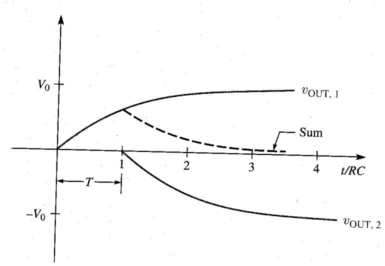
Response to a rectangular pulse (continue)

- The circuit on the right was excited by a rectangular function $V_{in}(t)$. The V_{out} is shown in the plot below.

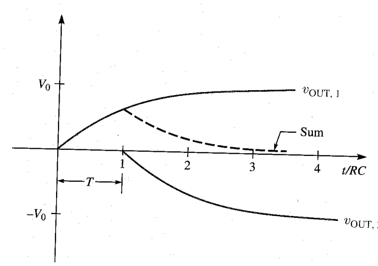


(a)

$V_{out,1}$ is response to V_1(previous slide)
$V_{out,2}$ is response to V_2



Response to a rectangular pulse (continue)

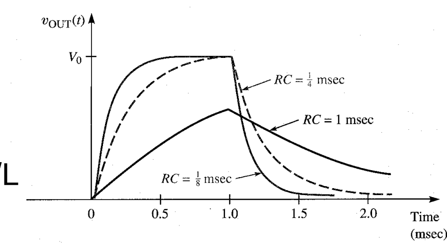


	$t < 0$	$0 < t < T$	$T < t$
Response to v_1	0	$V_0(1 - e^{-t/RC})$	$V_0(1 - e^{-t/RC})$
Response to v_2	0	0	$-V_0(1 - e^{-(t-T)/RC})$
Sum $= v_{OUT}$	0	$V_0(1 - e^{-t/RC})$	$V_0 e^{-t/RC}(e^{T/RC} - 1)$

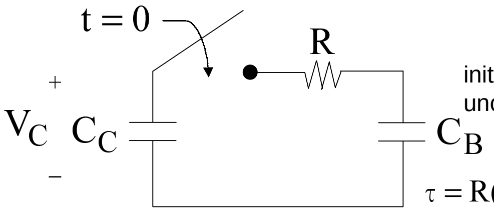
3 regions: t<0, 0<t<T, T<t, where T is the delay between v_1 and v_2.

Effect of τ (RC), the time constant?

Notice in LR circuit, τ is R/L



DIGITAL CIRCUIT EXAMPLE (Memory cell is read like this in DRAM)



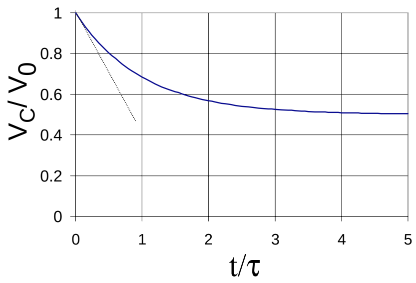
initially uncharged

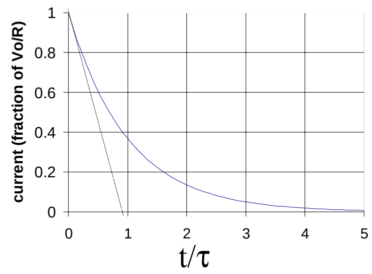
For simplicity, let $C_C = C_B$. If $V_C = V_0$, $t < 0$.

Find $V_C(t)$, $i(t)$, energy dissipated in R.

$$\tau = R\left(\frac{C_C C_B}{C_C + C_B}\right) = \frac{R\,C_C}{2} \quad \text{if } C_C = C_B$$

$$V_C(0^+) = V_0 \qquad V_C(\infty) = \frac{1}{2}V_0 \quad (\text{conservation of Q for } C_C = C_B)$$





Summary

- Transient response is the behavior of a circuit in response to a change in input. The transient response dies away in time. What is left after the transient has died away is steady state response.
- Voltage across a capacitor can NOT change suddenly. The current thru an inductor can not change suddenly.
- In dc steady state, the current thru a capacitor and the voltage across a inductor must be zero.
- For first order circuits, transient voltages and currents are of the form $A + B e^{-t/\tau}$, τ is the time constant. A and t are found by substituting the $A + B e^{-t/\tau}$ into the circuit equation (node/loop equations). The B is found from the initial condition.
- The response to a rectangular pulse is the sum of response to positive and negative going step inputs. The form of the output pulse depends on whether the duration of the input pulse (T) is long or short compared with the time constant τ.