Thévenin & Norton Equivalent Circuits

Lecture 5 review:

- Voltage divider & current divider
- Equivalent circuits (Thévenin & Norton equivalents)

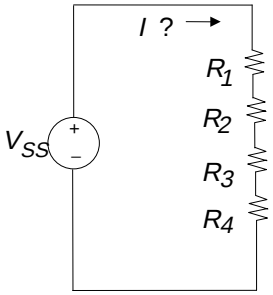
Today: (3.3, 3.4)

- Power Calculations
- Starting your car (model)

RESISTORS IN SERIES

(Here its more convenient to use KVL than node analysis)

Circuit with several resistors in series – Can we find an equivalent resistance?



- KCL tells us same current flows through every resistor
- KVL tells us

$$I \cdot R_1 + I \cdot R_2 + I \cdot R_3 + I \cdot R_4 = V_{SS}$$

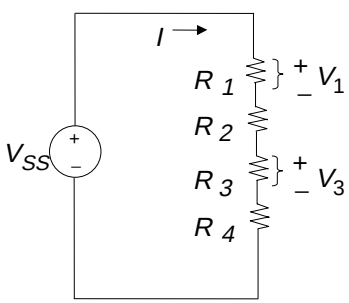
- Clearly,

$$I = V_{SS}/(R_1 + R_2 + R_3 + R_4)$$

☞ Thus, equivalent resistance of resistors in series is the simple sum

GENERALIZED VOLTAGE DIVIDER

Circuit with several resistors *in series*



- We know

$$I = V_{SS}/(R_1 + R_2 + R_3 + R_4)$$

- Thus,

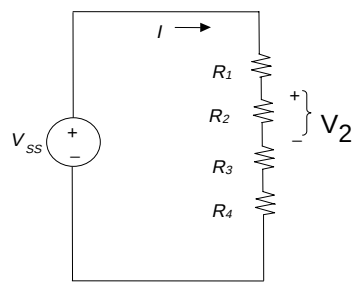
$$V_1 = \frac{R_1}{R_1 + R_2 + R_3 + R_4} \cdot V_{SS}$$

and

$$V_3 = \frac{R_3}{R_1 + R_2 + R_3 + R_4} \cdot V_{SS}$$

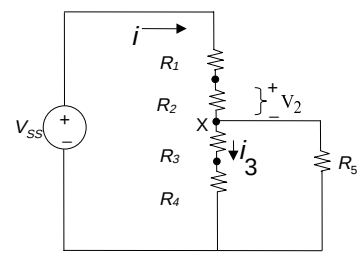
etc.. etc..

WHEN IS VOLTAGE DIVIDER FORMULA CORRECT?



$$V_2 = \frac{R_2}{R_1 + R_2 + R_3 + R_4} \times V_{SS}$$

Correct if nothing else connected to nodes



$$V_2 \neq \frac{R_2}{R_1 + R_2 + R_3 + R_4} \times V_{SS}$$

because R_5 removes condition of resistors in series – i.e. $i_3 \neq I$

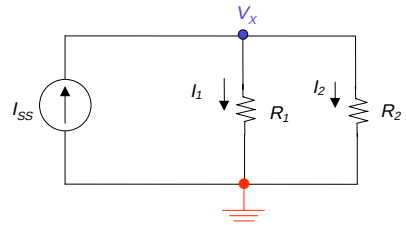
What is V_2?

Answer: $\dfrac{R_2}{R_1 + R_2 + R_5 \| (R_3 + R_4)} \times V_{SS}$

RESISTORS IN PARALLEL

1 Select Reference Node

2 Define unknown node voltages



Note: $I_{ss} = I_1 + I_2$, i.e.,

$$I_{SS} = \frac{V_X}{R_1} + \frac{V_X}{R_2} \Rightarrow V_X = I_{SS} \cdot \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = I_{SS} \cdot \frac{R_1 R_2}{R_1 + R_2}$$

RESULT 1 EQUIVALENT RESISTANCE: $R_{\|} \equiv R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$

RESULT 2 CURRENT DIVIDER:

$$I_1 = \frac{V_X}{R_1} = I_{SS} \times \frac{R_2}{R_1 + R_2}$$

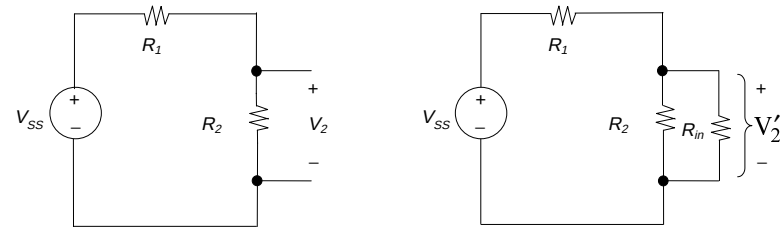
$$I_2 = \frac{V_X}{R_2} = I_{SS} \times \frac{R_1}{R_1 + R_2}$$

REAL VOLTMETERS

Concept of "Loading" as Application of Parallel Resistors

How is voltage measured? Modern answer: Digital multimeter (DMM)

Problem: Connecting leads from voltmeter across two nodes changes the circuit. The voltmeter is characterized by how much current it draws at a given voltage → "voltmeter input resistance," R_{in}. Typical value: 10 MΩ



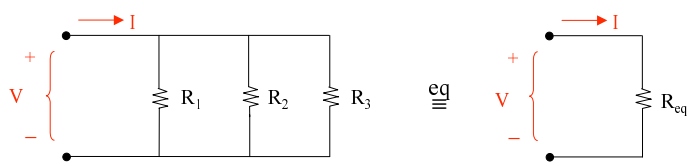
$$V_2 = V_{SS}\left[\frac{R_2}{R_1 + R_2}\right] \qquad V_2' = V_{SS}\left[\frac{R_2 \| R_{in}}{R_2 \| R_{in} + R_1}\right]$$

Example: $V_{SS} = 10\text{V}, R_2 = 100\text{K}, R_1 = 900\text{K} \Rightarrow V_2 = 1\text{V}$

But if $R_{in} = 10\text{M}, V_2' = 0.991\text{V}$, a 1% error

GENERALIZED PARALLEL RESISTORS

What single resistance R_{eq} is equivalent to three resistors in parallel?



$$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \equiv \frac{V}{R_{eq}} \Rightarrow R_{eq} \equiv R_1 \| R_2 \| R_3 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

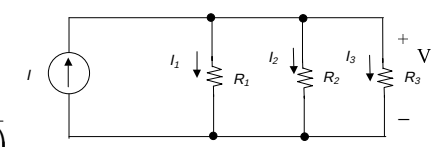
Note the simplicity if we use conductance instead of resistance

$$G_1 \equiv \frac{1}{R_1}, \text{etc.}, G_{eq} \equiv \frac{1}{R_{eq}}$$

Then, $G_{eq} = G_1 + G_2 + G_3$ ADD CONDUCTANCES IN PARALLEL

GENERALIZED CURRENT DIVIDER

Current splits among M resistors in parallel



Formal approach:

$$V = \frac{I}{\left(\frac{1}{R_1}\right) + \left(\frac{1}{R_2}\right) + \left(\frac{1}{R_3}\right)}$$

$$I_3 = \frac{V}{R_3} = I\left[\frac{1/R_3}{1/R_1 + 1/R_2 + 1/R_3}\right] \qquad \text{Note } I_3 = I\left[\frac{G_3}{G_1 + G_2 + G_3}\right]$$

$$I_3 = I\left[\frac{R_1 R_2}{R_2 R_3 + R_1 R_3 + R_1 R_2}\right] = I\left[\frac{R_1 R_2/(R_1 + R_2)}{R_1 R_2/(R_1 + R_2) + R_3}\right]$$

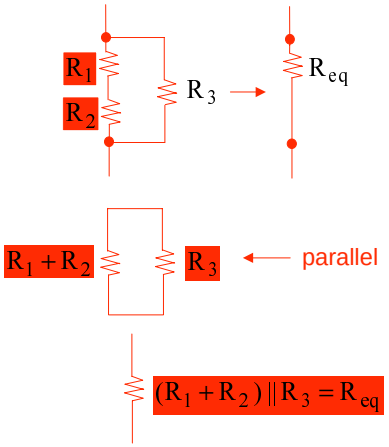
Can we get this result by inspection?

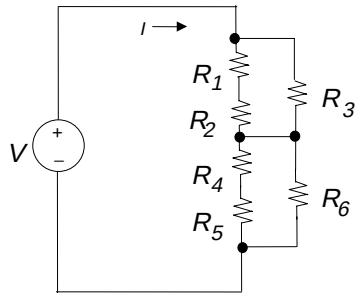
YES – Consider $R_1 \| R_2$ as "one equivalent resistor"

$$\text{Then } I_3 = \frac{R_1 \| R_2}{R_1 \| R_2 + R_3} \times I \qquad = \frac{R_1 R_2/(R_1 + R_2)}{R_1 R_2/(R_1 + R_2) + R_3} \times I$$

IDENTIFYING SERIES AND PARALLEL COMBINATIONS

Use series/parallel equivalents to simplify a circuit before starting KVL/KCL



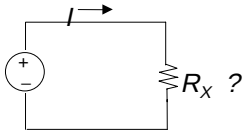


$R_1 = R_2 = 10\,K$

$R_3\ R_3 = 20\,K$

$R_4 = 5 = 5\,K$

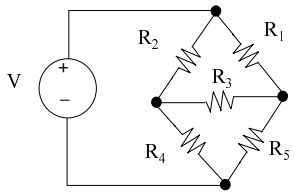
$R_6\ R_6 = 10\,K$



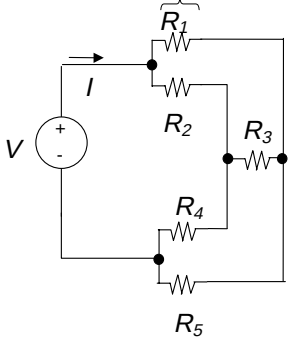
$$R_X = (R_1 + R_2) \| R_3 + (R_4 + R_5) \| R_6$$
$$= 15\,K$$

IDENTIFYING SERIES AND PARALLEL COMBINATIONS (cont.)

Some circuits *must* be analyzed (not amenable to simple inspection)



R_1 and R_2 are **not** in ||



R_1 and R_5 are **not** in series

Special cases:
$R_3 = 0$ **OR** $R_3 = \infty$

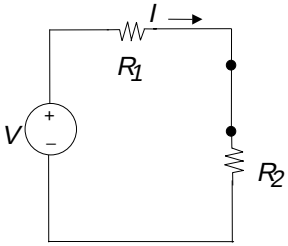
Example: $R_3 = 0 \Rightarrow R_1 \| R_2$; $R_4 \| R_5$ in series; $R_{eq} = R_1 \| R_2 + R_4 \| R_5$

OR IF $R_3 = \infty \Rightarrow (R_1 + R_5) \| (R_2 + R_4)$

MEASURING CURRENT

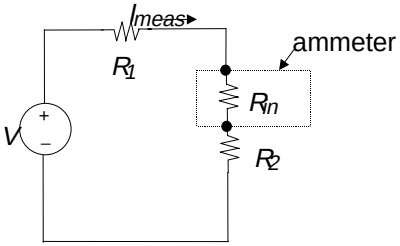
Insert DMM (in current measurement mode) into circuit. But ammeters disturb the circuit. (Note: Ammeters are characterized by their "ammeter input resistance," R_{in}. Ideally this should be very low. Typical value (in mA range) 1Ω.)

Potential measurement error due to non-zero input resistance:



undisturbed circuit

$$I = \frac{V_1}{R_1 + R_2}$$



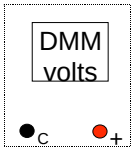
with ammeter

$$I_{meas} = \frac{V_1}{R_1 + R_2 + R_{in}}$$

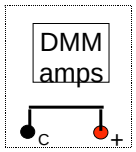
Example V = 1 V: R1 + R2 = 1 KΩ , Rin = 1Ω

$$I = 1\,mA \quad , \quad I_{meas} = \frac{1}{1K + 1\Omega} \cong 0.999\,mA \quad (0.1\%\ \text{error})$$

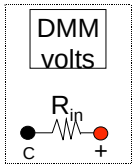
IDEAL AND NON-IDEAL METERS



IDEAL



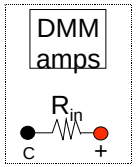
IDEAL



MODEL OF REAL DIGITAL VOLTMETER

Note: R_{in} may depend on range

R_{in} typically $\tilde{>}$ 10 MΩ

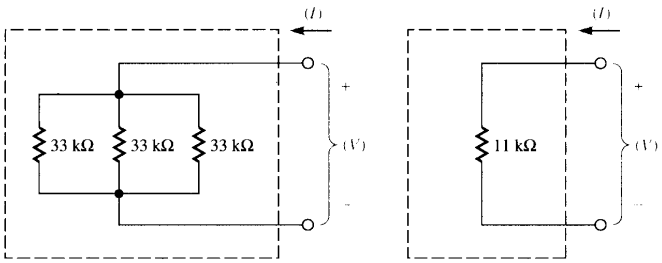


MODEL OF REAL DIGITAL AMMETER

Note: R_{in} usually depends on current range

R_{in} typically $\tilde{<}$ 1 Ω

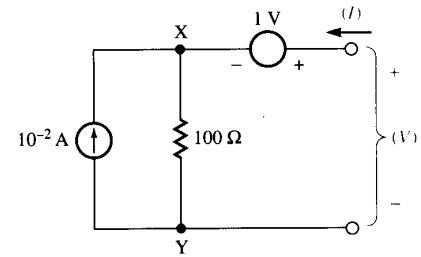
Equivalent Circuits



Circuits on the left side is equivalent to the circuit on the right. Both circuits have a V-I curve equivalent to a 11kΩ resistor.

Composite Circuit Element

Find the I-V relationship of the circuit on the right.



KCL equation at node X:

$$10^{-2} + \frac{V_Y - V_X}{100} + I = 0$$

$$V_X = V_Y + 100I + 1$$

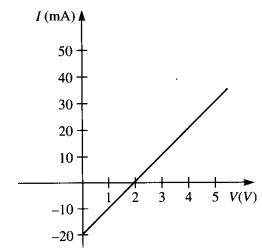
V is equal to: $V = V_{XY} + 1 = V_X - V_Y + 1$

Substitute 2nd eq to 3rd eq:

$$V = (V_Y + 100I + 1 - V_Y) + 1 = 100I + 2$$

or

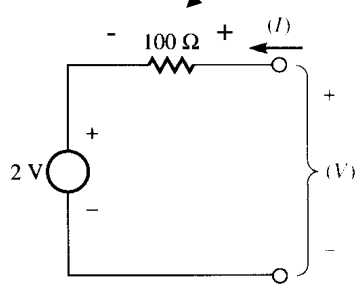
$$I = 10^{-2}V - 20\times10^{-3}$$



Example

Calculate the I-V relationship of the circuit.

Associated Ref. Direction



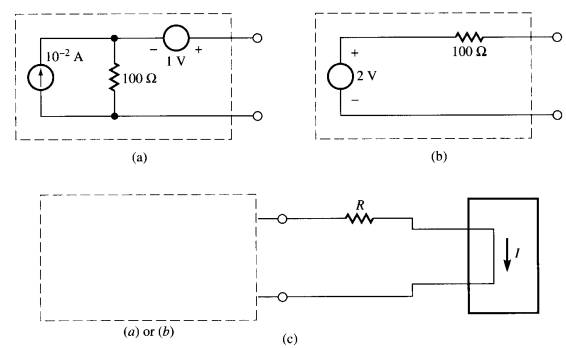
Write a KVL equation for this loop:

$$+V - 2 - 100I = 0$$

$$\Rightarrow \quad I = 10^{-2}V - 20\times10^{-3}$$

∴This circuit has an identical I-V relationship as the previous one ⇒ This circuit is equivalent to the previous circuit.

Equivalent circuit

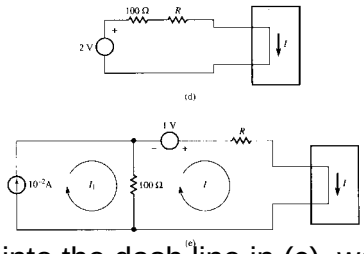


From the previous slide, ckt (a) and ckt (b) are equivalent. Let's check:

Substitute ckt (b) into the dash line in (c), we get ckt (d). We can solve this by Ohm's law:

$$2 = I(100 + R) \quad \Rightarrow \quad I = \frac{2}{100 + R} \quad \text{(from subckt (b))}$$

Equivalent circuit example (cont.)



Substitute ckt (c) into the dash line in (c), we get ckt (e). Here we use loop analysis. 2 mesh current I_1 and I. But I_1 is already known. It is equal 10^{-2}A.

The next loop equation is: $100(I - I_1) - 1 + IR = 0$

Solve for I yields: $I = \frac{2}{100 + R}$ (subckt (a)) (ckts (d) & (e) result in identical I)

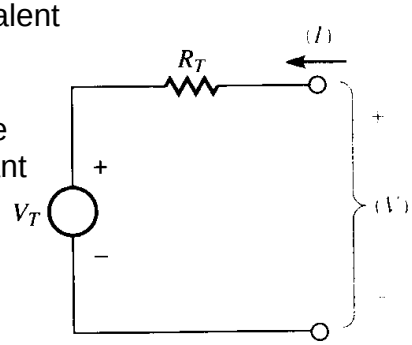
∴ The operation of the remainder of a ckt is unaffected when a subcircuit is replaced by its equivalent.

Equivalent circuits

- Two type of equivalent circuits
  - Thévenin equivalent circuits
  - Norton equivalent circuits

Thévenin Equivalent Circuits

General form of Thévenin equivalent is shown in the ckt on right.



V_T is called the Thévenin voltage
R_T is called the Thévenin resistant
By KVL around the loop:

$$V - V_T - R_T I = 0$$

Simplifies into:

$$I = \frac{V}{R_T} - \frac{V_T}{R_T}$$

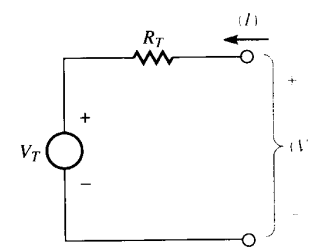
With $1/R_T$ as the slope in the I-V graph and $-V_T/R_T$ as the "y" intercept c. (Remember y=mx+c?)

Thévenin equivalent

In order to find the Thévenin equivalent of a circuit, we need to find V_T and R_T . The next question is how to find V_T and R_T.

When this circuit is open circuit, I = 0.

$$I = \frac{V}{R_T} - \frac{V_T}{R_T} \qquad (1)$$



becomes

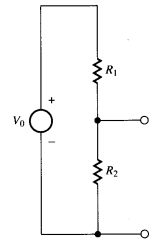
$$0 = \frac{V_{OC}}{R_T} - \frac{V_T}{R_T} \quad \Rightarrow \quad \boxed{V_T = V_{OC}}$$

We can compute the R_T by using the I_{SC} (short circuit current):

$$I_{SC} = \frac{0}{R_T} - \frac{V_T}{R_T} \quad \Rightarrow \quad \boxed{R_T = -\frac{V_T}{I_{SC}} = -\frac{V_{OC}}{I_{SC}}}$$

Thévenin equivalent example

Find the Thévenin equivalent circuit.



We can solve this by 2 steps:
(1) Find V_T by finding V_{OC}
(2) Find R_T by find I_{SC}

V_{OC} can be found by inspection (it is a voltage divider):

$$V_T = V_{OC} = V_0 \frac{R_2}{R_1 + R_2} \qquad \text{(a)}$$

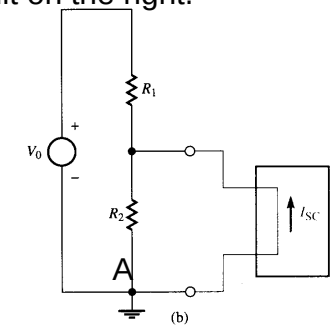
Thévenin equivalent example cont.

Next we calculate I_{SC} as shown in circuit on the right.

Apply KCL at node A:

$$\frac{V_0 - 0}{R_1} + \frac{0 - 0}{R_2} + I_{SC} = 0 \quad \Rightarrow \quad I_{SC} = -\frac{V_0}{R_1} \quad \text{(b)}$$

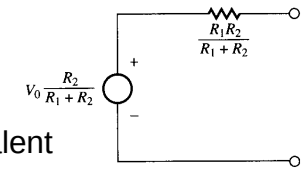
From previous slide, $R_T = -\dfrac{V_{OC}}{I_{SC}}$ (c)



(b)

Substitute (b) and (a) into (c)

$$R_T = V_{OC}\left(\frac{R_1}{V_0}\right) = V_0\left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{R_1}{V_0}\right) = \frac{R_1 R_2}{R_1 + R_2}$$



(c)

∴ Circuit © shows a Thévenin equivalent of the original circuit.

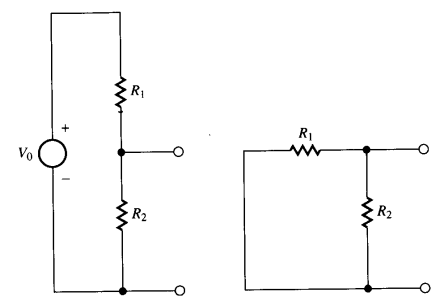
Thévenin equivalent (alternative method)

There is an alternative method for finding R_T:

- Locate all independent voltage & current sources inside the subcircuit whose equivalent is to be found.
- Replace all independent voltage sources by short circuits
- Replace all independent current sources by open circuits
- Compute the resistance between the 2 terminals

Thévenin equivalent (alternative method) example

Find the R_T by the alternative method:

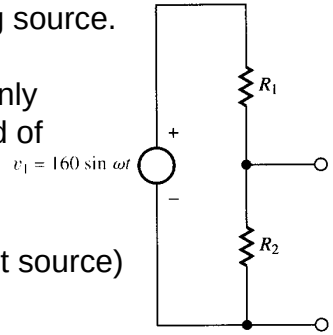


- Replace the voltage source by short circuit (current source by open circuit).
- Measure the resistance across the 2 terminals.
- The modified circuit becomes the circuit on the right.

$R_T = R_1 \parallel R_2$

Thévenin equivalent (time varying source)

This is identical to the previous example except that it has a time varying source.



The method is the same. The only difference is the source. Instead of V_0, we use V_1 as source:

$$V_T = V_{OC} = V_0 \frac{R_2}{R_1 + R_2} \quad \text{(time invariant source)}$$

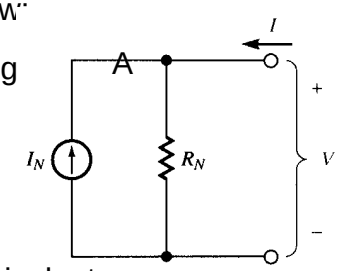
$$V_T = V_{OC} = V_1 \frac{R_2}{R_1 + R_2} = 160 \sin(\omega t) \frac{R_2}{R_1 + R_2} \quad \text{(time varying source)}$$

$$R_T = V_{OC}\left(\frac{R_1}{V_1}\right) = V_1\left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{R_1}{V_1}\right) = \frac{R_1 R_2}{R_1 + R_2} \quad \text{(time varying source)}$$

In this case, RT does not depend time, thus, the same.

Norton Equivalent Circuits

Principle of the Norton equivalent circuit is similar to that of the Thevenin equivalent circuit. Consider the general form of Norton equivalent circuit below:



We can find the I-V equation by applying KCL at node A:

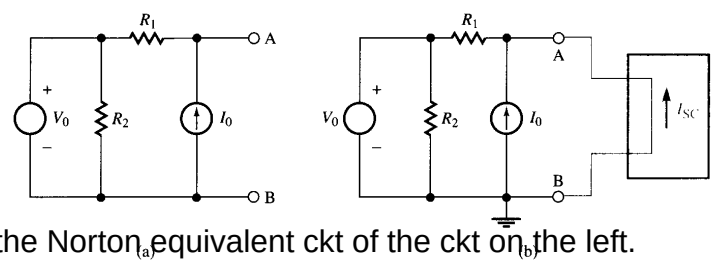
$$I - \frac{V}{R_N} + I_N = 0 \quad \Rightarrow \quad I = \frac{V}{R_N} - I_N \qquad (1)$$

Next we need to find I_N and R_N.
Use similar strategy as in Thevenin equivalent.
First short circuit the 2 terminals yields (I becomes I_{SC}, V=0 in eq. 1): $I_{SC} = \frac{0}{R_N} - I_N \quad \Rightarrow \quad \boxed{I_N = -I_{SC}}$

Next we open circuit the terminals (I=0, V=V_{OC} in eq. 1):

$$0 = \frac{V_{OC}}{R_N} - I_N \quad \Rightarrow \quad \boxed{R_T = -\frac{V_{OC}}{I_{SC}}}$$

Norton equivalent circuits (example)



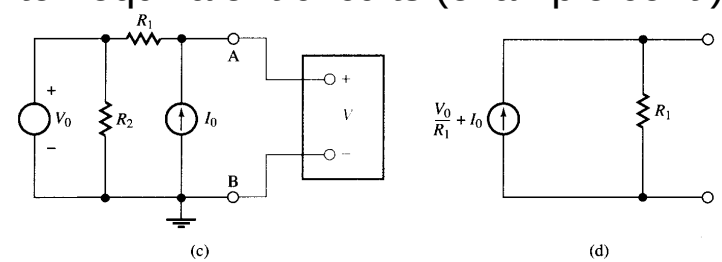
Find the Norton equivalent ckt of the ckt on the left.

First, find I_{SC} (the current indicated by an ideal ammeter connected to the terminals A and B.)

Apply KCL at node A in the right ckt : $\frac{V_0 - 0}{R_1} + I_0 + I_{SC} = 0$

or $I_{SC} = -\frac{V_0}{R_1} - I_0$

Norton equivalent circuits (example cont.)



Next, find V_{OC} (from ideal voltmeter connected to terminals A&B)

Apply KCL at node A (assume open circuit, I=0) (see circuit c):

$$\frac{V_0 - V_{OC}}{R_1} + I_0 = 0 \quad \text{or} \quad V_{OC} = I_0 R_1 + V_0$$

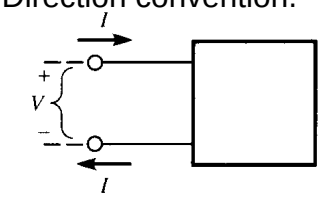
Use V_{OC} and I_{SC}, we can find I_N and R_N ($I_N = -I_{SC}$, $R_T = -\frac{V_{OC}}{I_{SC}}$):

$$I_N = \frac{V_0}{R_1} + I_0 \quad \text{and} \quad R_N = R_1$$

Power Calculations

Why?
- To find out how much power is being delivered to (from) some device, e.g. loud speaker.
- Alternative, it may be undesirable to delivered power to some part of the circuit, because of the heat it generates.
- Basic idea is addressed in the first lecture.

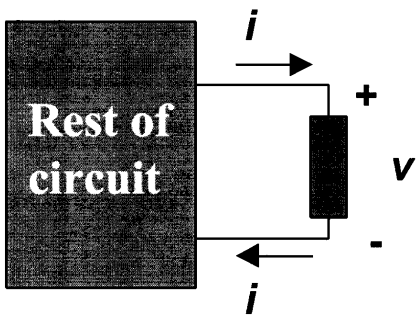
How?
Use the Associated Reference Direction convention.



Associated Reference Directions

- It is convenient to define the current through a circuit element as positive when entering the terminal associated with the + reference for voltage

For positive current and positive voltage, positive charge "falls down" a potential "drop" in moving through the circuit element: it absorbs power.



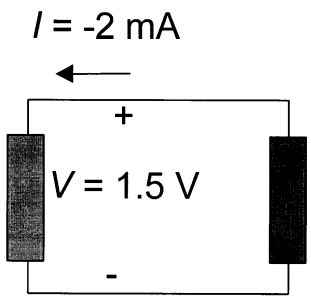
Power Definitions

- P = VI > 0 corresponds to the element absorbing power
  - How can a circuit element absorb power?
- By converting electrical energy into heat (resistors in toasters), light (light bulbs), acoustic energy (speakers); by storing energy (charging a battery)
- Negative power - releasing power to the rest of the circuit.

Conservation of Power

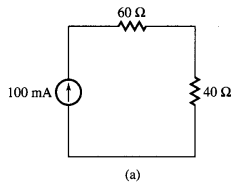
- Sum of the power absorbed by all circuit element must be zero.
- Concept: circuit elements are used to model all modes of energy conversion (heat, sound, batteries, voltage generators, etc.)
- Simple example:

Power released (VI <0) by the element on the left equals to the power absorbed by the element on the right.



Example of Power Flowing into Current Source

What is the power flow into the current source in the circuit on the right?

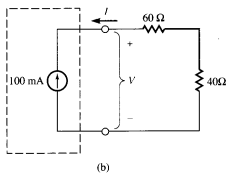


Put an imaginary box enclosing the current source and apply the associated reference direction (ARD) to the current source.

(1) Assume V convention on the right, ARD dictates that current has to go into + side of the terminals.

By KVL: V = 100mA×(60 Ω +40Ω) = 10V

By KCL: I = -100mA

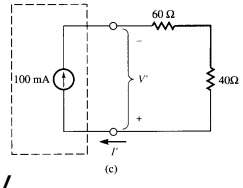


The power entering the box (in this case, current source) is:

Power = VI = 10V × -100mA = **-**1 W ∴1W is leaving the box

Example of Power Flowing into Current Source (cont)

(2) Assume V' convention on the right, ARD dictates that current has to go into + side of the terminals.



By KVL: V' = 100mA×(60 Ω +40Ω) = -10V

By KCL: I' = 100mA

The power entering the box (in this case, current source) is:

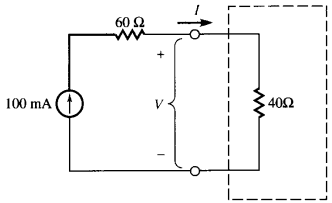
Power = V'I' = -10V × 100mA = **-**1 W ∴1W is leaving the box

Conclusions: V convention does not affect the results. Both conventions leads to the same conclusion that 1W is leaving the box or current source.

Example of Power Flowing into resistor

Find the power entering the 40 Ω resistor as shown in the circuit.

By associate reference direction convention. the V and I are defined.



By inspection I = 100mA

$$V = I \times 40\Omega = 4V$$

P = I × V = 100mA × 4V = 0.4W

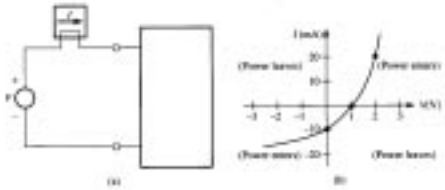
∴0.4W is entering the box (or 40Ω resistor).

(Using the similar strategy, it is found that 0.6W is entering the 60Ω resistor.)

Experimental Measurement of Power using a Volt Source and a Ammeter

The box in the right contains an unknown circuit, experimentally it is found that the I-V diagram as shown.

(how? Set the voltage, and measure the current by the ammeter)



V (set)	I (measured)	P = V I (compute)
-3V	-18mA	+54mW (power entering)
-2V	-16mA	+32mW
-1V	-14mA	+14mW
0	-10mA	0 (no power transfer)
1V	-2mA	-2mW (power leaving)
2V	+20mA	+40mW
3V	+400mA	+1.2W

∴ In the first and the third quadrant of the I-V curve, power is entering.
In the second and the fourth quadrant, power is leaving.

Special cases

If a voltage exists across a resistor R,

Power dissipated in the resistor = VI = V (V/R) =V^2/R
(Ohm's law applies in this case)

If a current flow, I, through a resistor R,

Power dissipated in the resistor = VI = (IR) I =I^2R
(Ohm's law also applies in this case)

Instantaneous Power and Average Power

When V and I are function of time, we write as v(t) and i(t), then instantaneous power entering the box is:

$P(t) = v(t)i(t)$ (instantaneous power, power as function of t)

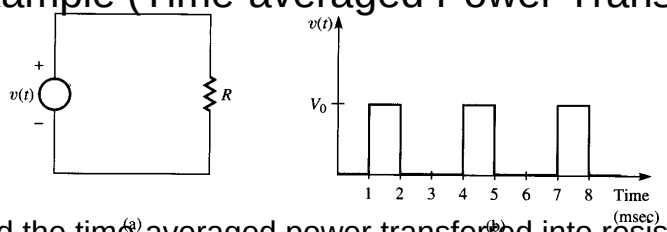
This is particularly useful to compute the maximum power received or delivered at any instant.

$P_{AV} = \frac{1}{T}\int_0^T v(t)i(t)dt$ (time-averaged power)

This is particularly useful for computing the trend, the average power received or delivered.

Does DC current have instantaneous or time-average power?

Example (Time-averaged Power Transferred)



Find the time-averaged power transferred into resistor R, if the time varying voltage source *v(t)* is as shown on the right.

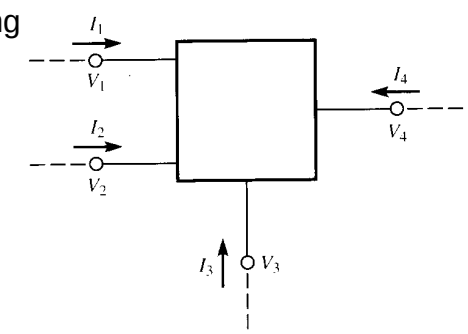
Observation: (1) v(t) is a square wave (max at V_0, min at 0)
i(t) is a _____wave (max at , min at)
Instantaneous power = v(t)i(t) = or
(2) Period from 0ms to 3ms, or 1ms to 4ms, or 2ms to 5ms.

$$P_{AV} = \frac{1}{0.003-0}\int_0^{0.003} v(t)i(t)dt$$

$$P_{AV} = \frac{1}{0.003}\left(\int_0^{0.001} 0dt + \int_{0.001}^{0.002}\frac{V_0^2}{R}dt + \int_{0.002}^{0.003} 0dt\right) = \frac{1}{0.003}\frac{V_0^2}{R}(0.001) = \frac{V_0^2}{3R}$$

Multi-terminal Elements

We dealt with circuit element with 2 terminals in the preceding sections.



Here we have 4 terminals

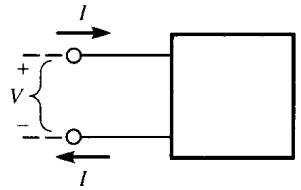
P = $V_1I_1+V_2I_2+V_3I_3+V_4I_4$

Notice the reference direction of I is toward the terminals, the terminals are labeled V_1, V_2, V_3, V_4.

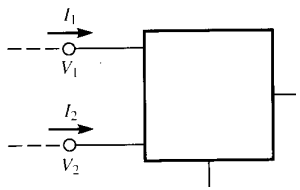
This can be generalized into any N terminals devices.

Example

Show that the 2 terminals circuit and formula is a special case to our 4 terminals circuit formula.



Is P=VI a special case of
$P= V_1I_1+V_2I_2+V_3I_3+V_4I_4$?
Generalize into a 2 terminals circuit:
$P= V_1I_1+V_2I_2$

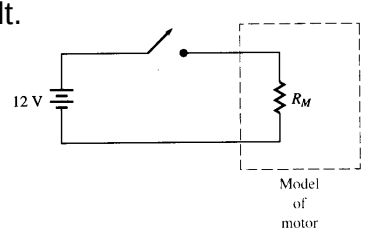


Comparing the 2 circuits,
$I=I_1$, $I=-1_2$
$P= V_1I_1+V_2I_2$ becomes $P= V_1I-V_2I=(V_1-V_2)I$
But $V=V_1-V_2$ (comparing the 2 circuits)

P=VI which is our original Power formula.
∴ P=VI is a special case of $P= V_1I_1+V_2I_2+V_3I_3+V_4I_4$ general formula.

Starting a Car

A starting motor of a car typical of a range of 20-40 Amp and 12 volt.



If we take 30Amp and 12 volt. The motor can be modeled as a equivalent resistance with value R_m=12/30 = 0.4Ω as shown in the diagram on the right.

Can we replace the bulky car battery by eight 1.5v AA size batteries in series? Why not? They are both 12V.

The battery model  needs to be replaced by a more accurate electrical model.

Battery model

We can model each battery as an ideal 12V voltage source. But this would not help us to understand the difference between eight 1.5V batteries in series and a 12V car battery.

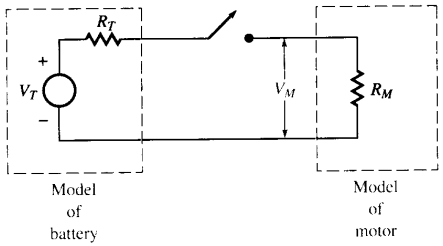
Next, we can model the battery (2 terminal voltage source) as a Thévenin equivalent circuit as shown below.

Both batteries have an identical V_T (open circuit voltage), so what is the difference between the two?

By voltage divider formula:

$$V_M = V_T \frac{R_M}{R_T + R_M}$$

Since V_T and R_M are the same for both batteries, the difference is in R_T.



Battery model (cont)

The typical car battery has a Thévenin resistance of 0.05 Ω.

$$V_M = 12\frac{0.4}{0.05+0.4} = 10.67V \qquad I = \frac{12}{0.05+0.4} = 26.67A$$

The typical AA size battery has a Thévenin resistance of 20 Ω.

$$V_M = 12\frac{0.4}{20+0.4} = 0.235V. \qquad I = \frac{12}{20+0.4} = 0.588A$$

That is not sufficient voltage and current to start a car.
This is no surprising because you don't expect AA size batteries to be sufficient to start a car!
Typically, the resistance (R_T) is a function of the cross section of the device (in this case, battery). The larger the cross section, the smaller the resistance.

Points To Remember:

• Equivalent circuits are circuits that cannot be distinguished from each other by measurements at their terminals. Often circuit analysis can be simplified if a portion of the circuit is replaced by a simpler equivalent. Two general families of equivalents exist for linear circuits: Thévenin equivalents and Norton equivalents.

• Power flow can be calculated from the expressions **P = VI** for two-terminal circuit elements and $\mathbf{P = \Sigma\, V_n\, I_n}$ for multi-terminal circuit element. However, it is essential that the signs of the various voltages and currents be stated correctly.

• If voltage and current vary, the quantity *v(t)i(t)* is known as the *instantaneous power*. The *time-averaged power* is the average over time of the instantaneous power.