Thévenin & Norton Equivalent Circuits

Lecture 4 review:

- Nodal analysis
- Loop analysis
- Series and parallel resistors

Today:

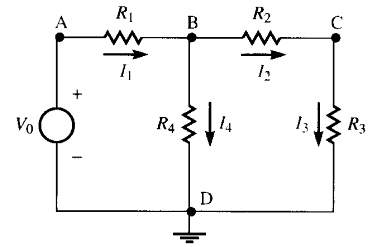
- Equivalent circuits (Thévenin & Norton equivalents)
- Series & parallel resistors (revisit & generalize)
- Voltage divider & current divider
- Real voltmeters & ammeters

Node Analysis

Goals: *Solve all unknown node voltages.*

How?

•*By writing equations expressing Kirchhoff's current law (KCL) for each node where the voltage is unknown.*



Node Analysis (cont.)

5 circuit elements - 4 resistors, 1 voltage source.

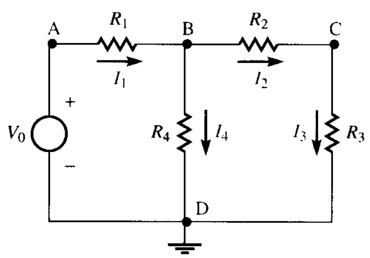
4 nodes:

2 unknown voltages: v_B,v_C

2 known voltages:

1 reference voltage: v_D

1 voltage source: $v_A=v_o$



4 branch currents: I_1, I_2, I_3, I_4

(but $I_2=I_3$) => 3 unknown currents

Apply KCL at node B:

$$\frac{V_0 - V_B}{R_1} + \frac{V_C - V_B}{R_2} + \frac{V_D - V_B}{R_4} = 0$$

Apply KCL at node C:

$$\frac{V_B - V_C}{R_2} + \frac{V_D - V_C}{R_3} = 0$$

But $V_D = 0$ in both cases.

Node Analysis

$$\frac{V_0 - V_B}{R_1} + \frac{V_C - V_B}{R_2} + \frac{-V_B}{R_4} = 0 \qquad \frac{V_B - V_C}{R_1} + \frac{-V_C}{R_3} = 0$$

2 equations and 2 unknowns.

Can be solved simultaneously for V_B & V_C

$$V_C = V_0 \cdot \frac{R_3 R_4}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$

$$V_B = V_0 \cdot \frac{R_4 (R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_4 + R_3 R_4}$$

$$\left.\begin{matrix} V_B = f(V_0) \\ V_C = f(V_0) \end{matrix}\right\}$$

$$\mathit{if} \quad \begin{cases} V_0 = 0 \\ V_B = V_C = 0 \end{cases}$$

What about the 4 branch currents?

Apply Ohm's law

$$I_1 = \frac{V_A - V_B}{R_1} = \frac{V_0 - V_B}{R_1} \quad I_2 = \frac{V_B - V_C}{R_2} \quad I_3 = \frac{V_C - V_D}{R_3} = \frac{V_C}{R_3} \quad I_4 = \frac{V_B - V_D}{R_4} = \frac{V_B}{R_4}$$

Node Analysis (nodal analysis)

1 Choose a Reference Node

2 Define unknown node voltages (those not fixed by voltage sources)

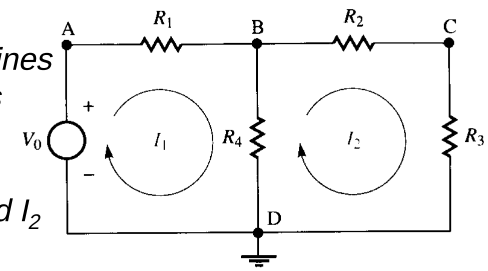
3 Write KCL at each unknown node, expressing current in terms of the node voltages (using the constitutive relationships of branch elements: **for resistor element use Ohm's law**)

4 Solve the set of equations (N equations for N unknown node voltages)

Loop Analysis

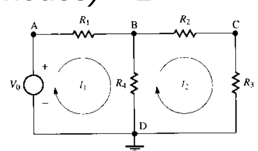
- *In node analysis (or nodal analysis), voltage is obtained first by using KCL, then current is found by using the V-I characteristic of the circuit element (for resistor element: Ohm's law).*
- *In loop analysis, current is obtained first by KVL, then voltage is found by using the V-I characteristic (for resistor element: Ohm's law).*

In loop analysis, one defines special current known as ***mesh currents.***
In this example, 2 mesh currents is defined, I_1 and I_2



Loop Analysis (Mesh Analysis)

The number of mesh currents = (Number of branches) - (Number of nodes) + 1
In this case, there are 4 nodes, 5 branches.
# of mesh = 5 - 4 + 1 = 2



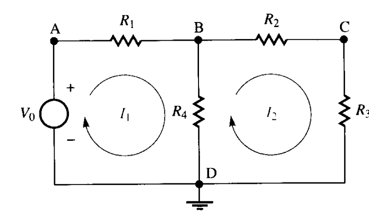
How to define mesh current?
A lot of freedom, except that every branch of the circuit have at least one mesh current flowing thru it.

The value of the mesh currents are now the unknown to be solved for. The number of equations = The number of unknown mesh current

The equations *are obtained by applying Kirchhoff Voltage law on the loops of the mesh currents.*

Loop Analysis

Assume I_1 and I_2 are defined as the mesh current. Notice the mesh current go thru the whole loop. R_4 has 2 mesh currents go thru it.



The voltage drop from A to B across R_1: $I_1 R_1$

The voltage drop from B to D across R4: $(I_1 - I_2)R_4$

Apply KVL in the loop ABDA:

$$I_1 R_1 + (I_1 - I_2)R_4 - V_0 = 0$$

Apply KVL in the loop BCDB:

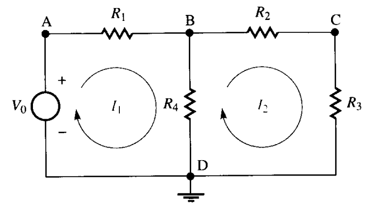
$$I_2 R_2 + I_2 R_3 + (I_2 - I_1)R_4 = 0$$

Loop Analysis

Solving these 2 equations simultaneously:

$$I_1 = V_0 \frac{R_2 + R_3 + R_4}{R_1R_2 + R_1R_3 + R_1R_4 + R_2R_4 + R_3R_4}$$

$$I_2 = V_0 \frac{R_4}{R_1R_2 + R_1R_3 + R_1R_4 + R_2R_4 + R_3R_4}$$



The individual branch current and voltage can be found from I_1 and I_2.

$$V_{CD} = V_C = I_2R_3 = V_0 \frac{R_4R_3}{R_1R_2 + R_1R_3 + R_1R_4 + R_2R_4 + R_3R_4}$$

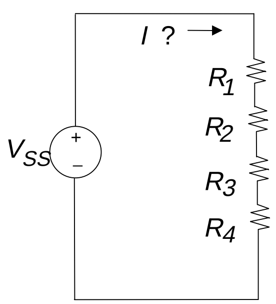
Loop Analysis (mesh analysis)

1. *Select the **proper number of mesh current** such that at least one mesh current passes through each branch.*
2. *(a) **Express voltage drop** across each element **as functions of known and unknown mesh currents**,*
   *(b) write equations stating that the sum of the voltage drop around closed path are zero. (**KVL**)*
3. ***Solve equations** obtained in step 2 simultaneously for unknown mesh current.*
4. *Obtain branch currents from the mesh current found in in step 3 and obtain desired node voltages from branch currents and the I-V relationship of the branches.*

RESISTORS IN SERIES

(Here its more convenient to use KVL than node analysis)

Circuit with several resistors in series – Can we find an equivalent resistance?



- KCL tells us same current flows through every resistor
- KVL tells us

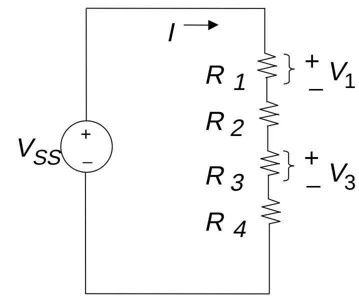
  $I \cdot R_1 + I \cdot R_2 + I \cdot R_3 + I \cdot R_4 = V_{SS}$
- Clearly,

  $I = V_{SS}/(R_1 + R_2 + R_3 + R_4)$

☞ Thus, equivalent resistance of resistors in series is the simple sum

GENERALIZED VOLTAGE DIVIDER

Circuit with several resistors *in series*



- We know

  $I = V_{SS}/(R_1 + R_2 + R_3 + R_4)$
- Thus,

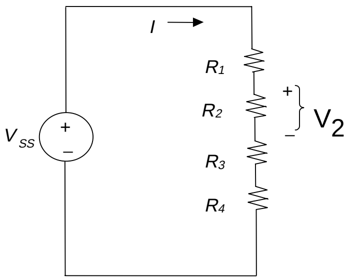
  $$V_1 = \frac{R_1}{R_1 + R_2 + R_3 + R_4} \cdot V_{SS}$$

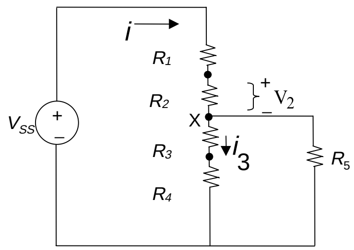
  and

  $$V_3 = \frac{R_3}{R_1 + R_2 + R_3 + R_4} \cdot V_{SS}$$

etc.. etc..

WHEN IS VOLTAGE DIVIDER FORMULA CORRECT?





$$V_2 = \frac{R_2}{R_1+R_2+R_3+R_4} \times V_{SS}$$

Correct if nothing else connected to nodes

$$V_2 \neq \frac{R_2}{R_1+R_2+R_3+R_4} \times V_{SS}$$

because R_5 removes condition of resistors in series – i.e. $i_3 \neq I$

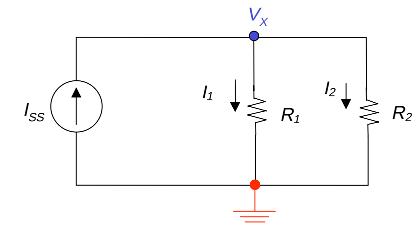
What is V_2?

Answer: $\dfrac{R_2}{R_1+R_2+R_5 \| (R_3+R_4)} \times V_{SS}$

RESISTORS IN PARALLEL

1 Select Reference Node

2 Define unknown node voltages



Note: $I_{ss} = I_1 + I_2$, i.e.,

$$I_{SS} = \frac{V_X}{R_1} + \frac{V_X}{R_2} \Rightarrow V_X = I_{SS} \cdot \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}} = I_{SS} \cdot \frac{R_1 R_2}{R_1+R_2}$$

RESULT 1 EQUIVALENT RESISTANCE: $R_{\|} \equiv R_1 \| R_2 = \dfrac{R_1 R_2}{R_1+R_2}$

RESULT 2 CURRENT DIVIDER:

$$I_1 = \frac{V_X}{R_1} = I_{SS} \times \frac{R_2}{R_1+R_2}$$

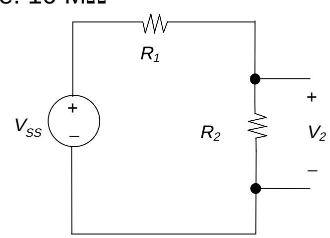
$$I_2 = \frac{V_X}{R_2} = I_{SS} \times \frac{R_1}{R_1+R_2}$$

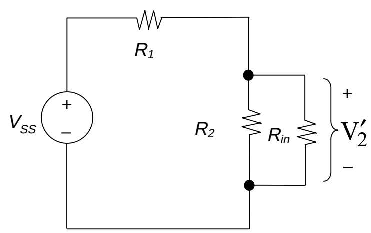
REAL VOLTMETERS

Concept of "Loading" as Application of Parallel Resistors

How is voltage measured? Modern answer: Digital multimeter (DMM)

Problem: Connecting leads from voltmeter across two nodes changes the circuit. The voltmeter is characterized by how much current it draws at a given voltage → "voltmeter input resistance," R_{in}. Typical value: 10 MΩ





$$V_2 = V_{SS}\left[\frac{R_2}{R_1+R_2}\right]$$

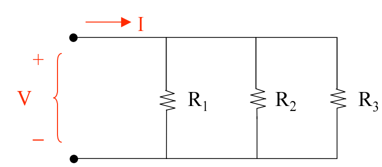
$$V_2' = V_{SS}\left[\frac{R_2 \| R_{in}}{R_2 \| R_{in} + R_1}\right]$$

Example: $V_{SS} = 10\text{V}, R_2 = 100\text{K}, R_1 = 900\text{K} \Rightarrow V_2 = 1\text{V}$

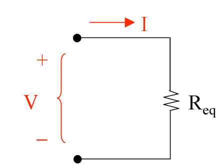
But if $R_{in} = 10\text{M}, V_2' = 0.991\text{V}$, a 1% error

GENERALIZED PARALLEL RESISTORS

What single resistance R_{eq} is equivalent to three resistors in parallel?



eq ≡



$$\boxed{I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} \equiv \frac{V}{R_{eq}}} \Rightarrow R_{eq} \equiv R_1 \| R_2 \| R_3 = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$$

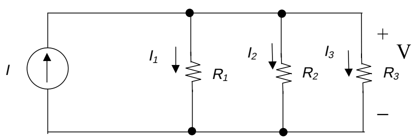
Note the simplicity if we use conductance instead of resistance

$$\boxed{G_1 \equiv \frac{1}{R_1}, \text{etc.}, G_{eq} \equiv \frac{1}{R_{eq}}}$$

Then, $\boxed{G_{eq} = G_1 + G_2 + G_3}$ ADD CONDUCTANCES IN PARALLEL

GENERALIZED CURRENT DIVIDER

Current splits among M resistors in parallel



Formal approach: $V = \dfrac{I}{\left(\frac{1}{R_1}\right)+\left(\frac{1}{R_2}\right)+\left(\frac{1}{R_3}\right)}$

$$I_3 = \frac{V}{R_3} = I\left[\frac{1/R_3}{1/R_1 + 1/R_2 + 1/R_3}\right] \qquad \text{Note } I_3 = I\left[\frac{G_3}{G_1+G_2+G_3}\right]$$

$$I_3 = I\left[\frac{R_1R_2}{R_2R_3 + R_1R_3 + R_1R_2}\right] \quad = I\left[\frac{R_1R_2/(R_1+R_2)}{R_1R_2/(R_1+R_2)+R_3}\right]$$

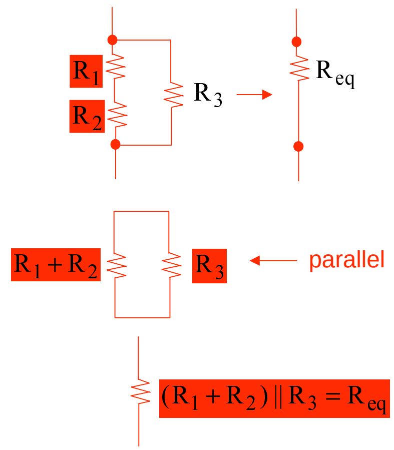
Can we get this result by inspection?

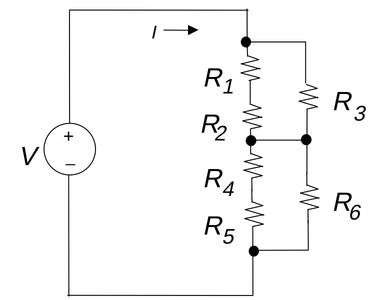
YES – Consider $R_1 || R_2$ as "one equivalent resistor"

Then $I_3 = \dfrac{R_1 || R_2}{R_1 || R_2 + R_3} \times I$ $\boxed{= \dfrac{R_1R_2/(R_1+R_2)}{R_1R_2/(R_1+R_2)+R_3} \times I}$

IDENTIFYING SERIES AND PARALLEL COMBINATIONS

Use series/parallel equivalents to simplify a circuit before starting KVL/KCL



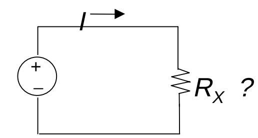


$R_1 = R_2 = 10\text{ K}$

$R_3 = 20\text{ K}$

$R_4 = 5 = 5\text{ K}$

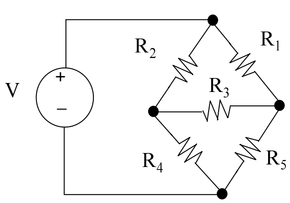
$R_6 = 10\text{ K}$



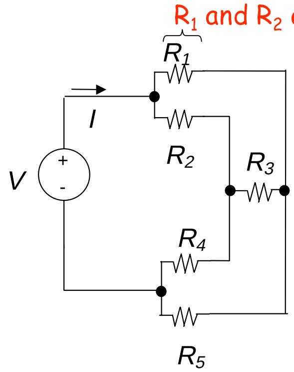
$$R_X = (R_1+R_2) || R_3 + (R_4+R_5) || R_6 = 15\text{ K}$$

IDENTIFYING SERIES AND PARALLEL COMBINATIONS (cont.)

Some circuits *must* be analyzed (not amenable to simple inspection)



R_1 and R_2 are **not** in ||



R_1 and R_5 are **not** in series

Special cases:
$R_3 = 0$ **OR** $R_3 = \infty$

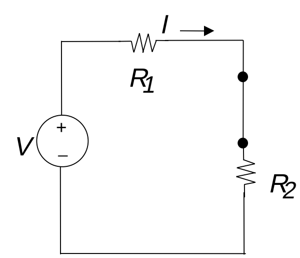
Example: $R_3 = 0 \Rightarrow R_1 || R_2$; $R_4 || R_5$ in series; $R_{eq} = R_1 || R_2 + R_4 || R_5$

OR IF $R_3 = \infty \Rightarrow (R_1 + R_5) || (R_2 + R_4)$

MEASURING CURRENT

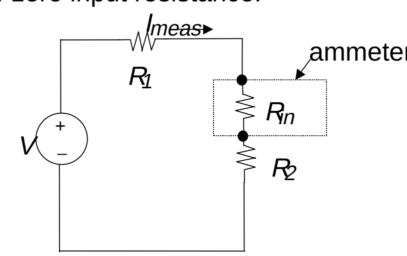
Insert DMM (in current measurement mode) into circuit. But ammeters disturb the circuit. (Note: Ammeters are characterized by their "ammeter input resistance," R_{in}. Ideally this should be very low. Typical value (in mA range) 1Ω.)

Potential measurement error due to non-zero input resistance:



undisturbed circuit

$$I = \frac{V_1}{R_1 + R_2}$$



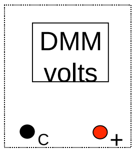
with ammeter

$$I_{meas} = \frac{V_1}{R_1 + R_2 + R_{in}}$$

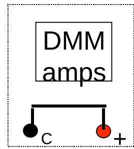
Example V = 1 V: R1 + R2 = 1 KΩ , Rin = 1Ω

$$I = 1\text{mA} \quad , \quad I_{meas} = \frac{1}{1\text{K} + 1\Omega} \cong 0.999\text{ mA} \quad (0.1\%\text{ error})$$

IDEAL AND NON-IDEAL METERS



IDEAL



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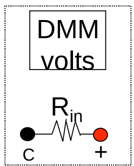
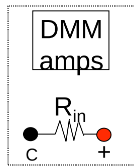


MODEL OF REAL DIGITAL VOLTMETER

Note: R_{in} may depend on range

R_{in} typically $\gtrsim$ 10 MΩ

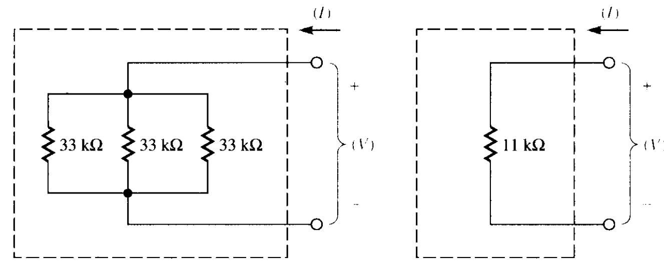


MODEL OF REAL DIGITAL AMMETER

Note: R_{in} usually depends on current range

R_{in} typically $\lesssim$ 1 Ω

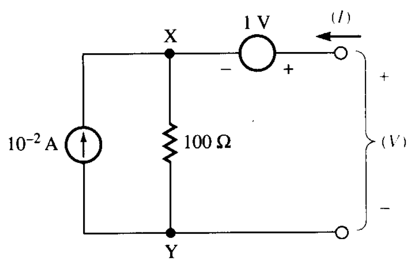
Equivalent Circuits



Circuits on the left side is equivalent to the circuit on the right. Both circuits have a V-I curve equivalent to a 11kΩ resistor.

Composite Circuit Element

Find the I-V relationship of the circuit on the right.



KCL equation at node X:

$$10^{-2} + \frac{V_Y - V_X}{100} + I = 0$$

$$V_X = V_Y + 100I + 1$$

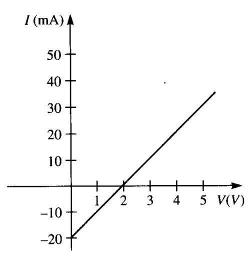
V is equal to: $V = V_{XY} + 1 = V_X - V_Y + 1$

Substitute 2nd eq to 3rd eq:

$$V = (V_Y + 100I + 1 - V_Y) + 1 = 100I + 2$$

or

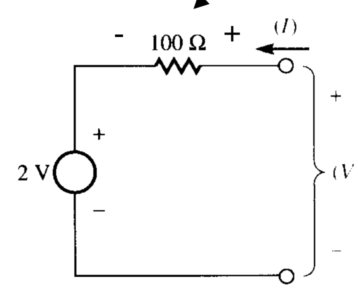
$$I = 10^{-2}V - 20\times 10^{-3}$$



Example

Calculate the I-V relationship of the circuit.

Associated Ref. Direction



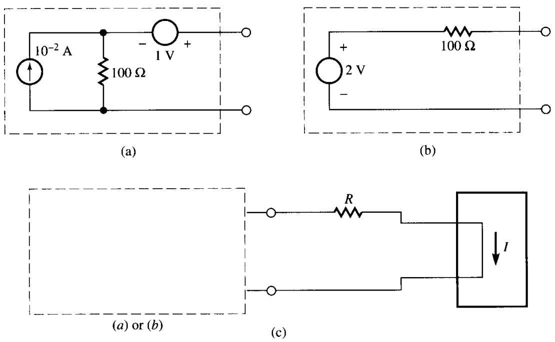
Write a KVL equation for this loop:

$$+V - 2 - 100I = 0$$

$$\Rightarrow \quad I = 10^{-2}V - 20\times 10^{-3}$$

∴This circuit has an identical I-V relationship as the previous one ⇒ This circuit is equivalent to the previous circuit.

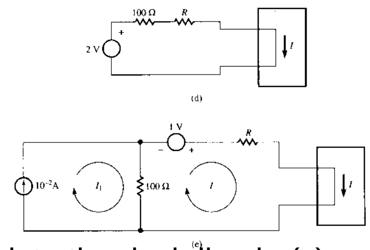
Equivalent circuit



From the previous slide, ckt (a) and ckt (b) are equivalent. Let's check:
Substitute ckt (b) into the dash line in (c), we get ckt (d). We can solve this by Ohm's law:

$$2 = I(100+R) \quad \Rightarrow \quad I = \frac{2}{100+R}$$ (from subckt (b))

Equivalent circuit example (cont.)



Substitute ckt (c) into the dash line in (c), we get ckt (e). Here we use loop analysis. 2 mesh current I_1 and I. But I_1 is already known. It is equal 10^{-2}A.

The next loop equation is: $100(I - I_1) - 1 + IR = 0$

Solve for I yields: $I = \frac{2}{100+R}$ (subckt (a)) (ckts (d) & (e) result in identical *I*)

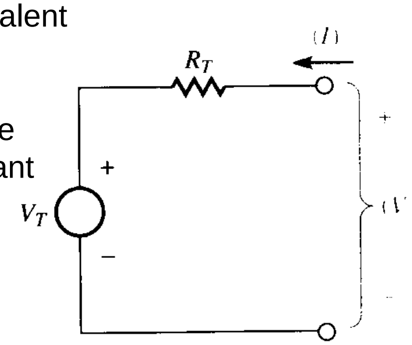
∴ The operation of the remainder of a ckt is unaffected when a subcircuit is replaced by its equivalent.

Equivalent circuits

- Two type of equivalent circuits
  - Thévenin equivalent circuits
  - Norton equivalent circuits

Thévenin Equivalent Circuits

General form of Thévenin equivalent is shown in the ckt on right.



V_T is called the Thévenin voltage
R_T is called the Thévenin resistant
By KVL around the loop:

$$V - V_T - R_T I = 0$$

Simplifies into:

$$I = \frac{V}{R_T} - \frac{V_T}{R_T}$$

With $1/R_T$ as the slope in the I-V graph and $-V_T/R_T$ as the "y" intercept c. (Remember y=mx+c?)

Thévenin equivalent

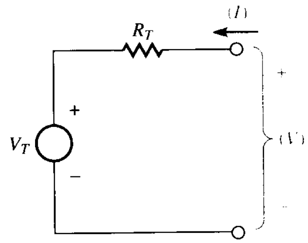
In order to find the Thévenin equivalent of a circuit, we need to find V_T and R_T . The next question is how to find V_T and R_T.

When this circuit is open circuit, I = 0.

$$I = \frac{V}{R_T} - \frac{V_T}{R_T} \qquad (1)$$

becomes

$$0 = \frac{V_{OC}}{R_T} - \frac{V_T}{R_T} \quad \Rightarrow \quad \boxed{V_T = V_{OC}}$$



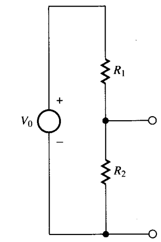
We can compute the R_T by using the I_{SC} (short circuit current):

$$I_{SC} = \frac{0}{R_T} - \frac{V_T}{R_T} \quad \Rightarrow \quad \boxed{R_T = -\frac{V_T}{I_{SC}} = -\frac{V_{OC}}{I_{SC}}}$$

Thévenin equivalent example

Find the Thévenin equivalent circuit.

We can solve this by 2 steps:
(1) Find V_T by finding V_{OC}
(2) Find R_T by find I_{SC}



V_{OC} can be found by inspection (it is a voltage divider):

$$V_T = V_{OC} = V_0 \frac{R_2}{R_1 + R_2} \qquad \text{(a)}$$

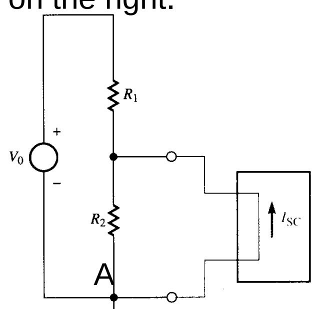
Thévenin equivalent example cont.

Next we calculate I_{SC} as shown in circuit on the right.

Apply KCL at node A:

$$\frac{V_0 - 0}{R_1} + \frac{0 - 0}{R_2} + I_{SC} = 0 \quad \Rightarrow \quad I_{SC} = -\frac{V_0}{R_1} \quad \text{(b)}$$

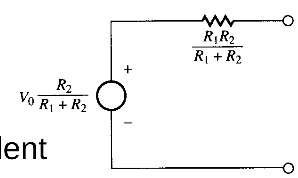
From previous slide, $R_T = -\dfrac{V_{OC}}{I_{SC}}$ (c)



(b)

Substitute (b) and (a) into (c)

$$R_T = V_{OC}\left(\frac{R_1}{V_0}\right) = V_0\left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{R_1}{V_0}\right) = \frac{R_1 R_2}{R_1 + R_2}$$



(c)

∴ Circuit © shows a Thévenin equivalent of the original circuit.

Thévenin equivalent (alternative method)

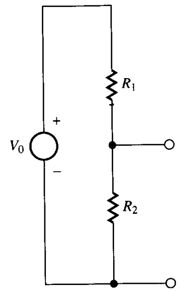
There is an alternative method for finding R_T:

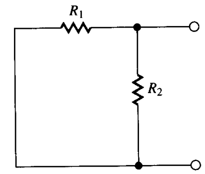
- Locate all independent voltage & current source inside the subcircuit whose equivalent is to be found.
- Replace all independent voltage sources by short circuits
- Replace all independent current sources by open circuits
- Compute the resistance between the 2 terminals

Thévenin equivalent (alternative method) example

Find the R_T by the alternative method:

- Replace the voltage source by short circuit (current source by open circuit).
- Measure the resistance across the 2 terminals.
- The modified circuit becomes the circuit on the right.



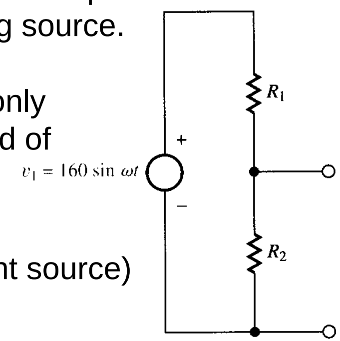


$R_T = R_1 \parallel R_2$

Thévenin equivalent (time varying source)

This is identical to the previous example except that it has a time varying source.

The method is the same. The only difference is the source. Instead of V_0, we use V_1 as source:



$$V_T = V_{OC} = V_0 \frac{R_2}{R_1 + R_2}$$ (time invariant source)

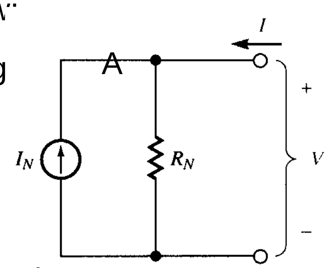
$$V_T = V_{OC} = V_1 \frac{R_2}{R_1 + R_2} = 160 \sin(\omega t) \frac{R_2}{R_1 + R_2}$$ (time varying source)

$$R_T = V_{OC}\left(\frac{R_1}{V_1}\right) = V_1\left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{R_1}{V_1}\right) = \frac{R_1 R_2}{R_1 + R_2}$$

(time varying source) In this case, RT does not depend time, thus, the same.

Norton Equivalent Circuits

Principle of the Norton equivalent circuit is similar to that of the Thevenin equivalent circuit. Consider the general form of Norton equivalent circuit below:



We can find the I-V equation by applying KCL at node A:

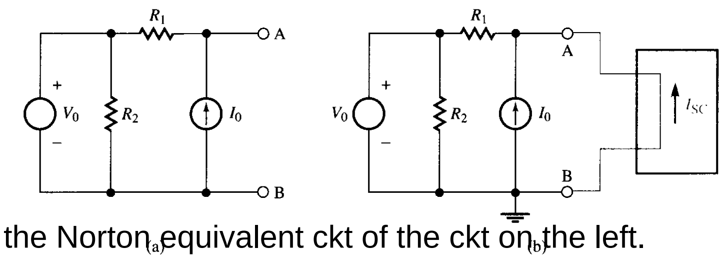
$$I - \frac{V}{R_N} + I_N = 0 \quad \Rightarrow \quad I = \frac{V}{R_N} - I_N \qquad (1)$$

Next we need to find I_N and R_N.
Use similar strategy as in Thevenin equivalent.
First short circuit the 2 terminals yields (I becomes I_{SC}, V=0 in eq. 1): $I_{SC} = \frac{0}{R_N} - I_N \quad \Rightarrow \quad \boxed{I_N = -I_{SC}}$

Next we open circuit the terminals (I=0, V=V_{OC} in eq. 1):

$$0 = \frac{V_{OC}}{R_N} - I_N \qquad \Rightarrow \qquad \boxed{R_T = -\frac{V_{OC}}{I_{SC}}}$$

Norton equivalent circuits (example)



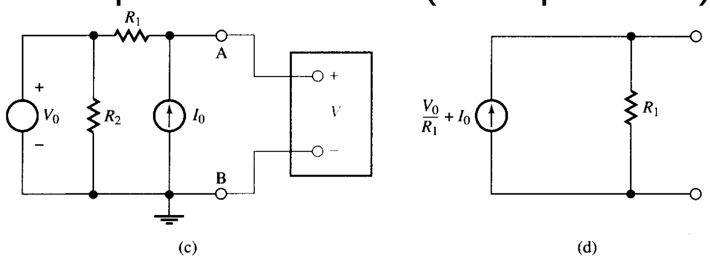
Find the Norton equivalent ckt of the ckt on the left.

First, find I_{SC} (the current indicated by an ideal ammeter connected to the terminals A and B.)

Apply KCL at node A in the right ckt : $\frac{V_0 - 0}{R_1} + I_0 + I_{SC} = 0$

or $I_{SC} = -\frac{V_0}{R_1} - I_0$

Norton equivalent circuits (example cont.)



Next, find V_{OC} (from ideal voltmeter connected to terminals A&B)

Apply KCL at node A (assume open circuit, I=0) (see circuit c):

$$\frac{V_0 - V_{OC}}{R_1} + I_0 = 0 \qquad \text{or} \qquad V_{OC} = I_0 R_1 + V_0$$

Use V_{OC} and I_{SC}, we can find I_N and R_N ($I_N = -I_{SC}$, $R_T = -\frac{V_{OC}}{I_{SC}}$):

$$I_N = \frac{V_0}{R_1} + I_0 \qquad \text{and} \qquad R_N = R_1$$