FACT 1) ELECTRICAL QUANTITIES IN PRESENCE OF SINUSOIDAL SOURCES WITH THE SAME FREQUENCY

In a linear system we can use the superposition theorem. Since a circuit is a linear system (even in presence of capacitors and inductors) every electrical quantity is the weighted sum of the sources that are present in the circuit.
If the source have all the form:

$$A\cos(\omega_0 t + \varphi)$$

where A and φ can change but ω_0 is always the same (isofrequency) then each quantity has the form

$$A\cos(\omega_0 t + \varphi)$$

where, of course, A an φ are in general different from the sources. This is true because we know that a quantity in the circuit is:

$$i_x(t) = \sum_i A_i \cos(\omega_0 t\ \varphi_i)$$

and that sume can be reduced to

$A'\cos(\omega_0 t + \varphi')$ by using Prostaferesi's formules.

Actually, since we are only interested in

A an φ (magnitude and phase) we could

use complex numbers to represent

quantities in this kind of situations.

We could use the bijection:

$$A\cos(\omega_0 t + \varphi) \longrightarrow Ae^{j\varphi}$$

This is the phasors method that you will

see in discussion session.

– LAPLACE TRASFORM APPLIED TO LINEAR, TIME INVARIANT CIRCUITS

To understand the trasform, let's look at an easy transformation.

Think of a world where we know how to do $\ln x$ and e^x, also we know how to sum real numbers. Unfortunately multiplication in this world is incredibly expensive: there is a tax for each multiplication you want to do.

We can use a trasformed domain.

The trasformation is the following:

$$\mathcal{L}: \mathbb{R}^+ \longrightarrow \mathbb{R} \quad \text{s.t.} \quad x \longrightarrow \ln x$$

it is always possible to find x by using an inverse tranformation

$$y \rightarrow e^y$$

So now if we have to multiply two numbers $x \cdot y$ we can work in the tranformed domain:

$$\ln(x \cdot y) = \ln x + \ln y = z$$

then $e^z = x \cdot y$ ← inverse tranform

the logaritm trasform products in sum.
Products in the original domain are sums in the trasformed domain

- LAPLACE TRANSFORM

It trasforms functions of t in functions of a complex variable $s=\alpha+j\omega$

$$\mathcal{L}[f(t)] = \lim_{T\to\infty}\int_0^T f(t)\,e^{-st}\,dt = F(s)$$

There exists an inverse trasformation

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\,e^{st}\,dt$$

where c is an appropriate real value.
This trasform has a lot of properties. We note that it is a linear trasformation

$$\mathcal{L}[A f_1(t) + B f_2(t)] = A F_1(s) + B F_2(s)$$

We are interested in the derivation and integration theorems:

$$\mathcal{L}\left[\frac{d f(t)}{dt}\right] = sF(s) - f(0)$$

Proof: using integration by parts

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt =$$

$$= f(\infty)\cdot 0 - f(0)e^0 + sF(s) = sF(s) - f(0) \quad \square$$

COROLLARY:

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = G(s)$$

$$\mathcal{L}\left[\frac{d}{dt}\int_0^t f(t)\,dt\right] = F(s) = sG(s) - \int_0^0 f(t)\,dt =$$

$$= sG(s) \Rightarrow G(s) = \frac{F(s)}{s}$$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \; ! \quad \square$$

1) derivative in time domain is multiplication by s in the s domain..
2) Integration in time domain is multiplication by $\frac{1}{s}$ in the s domain

- TOWARDS DEFINITION OF IMPEDANCE

For a capacitor:

$$i(t) = C\frac{dv(t)}{dt}$$

Transform each side of the equation:

$$I(s) = C\Big(sV(s) - v(0)\Big) = CsV(s) - Cv(0)$$

if $v(0) = 0 \Rightarrow I(s) = CsV(s) \Rightarrow \dfrac{V(S)}{I(S)} = \dfrac{1}{SC}$

For an inductor:

$$v(t) = L\frac{di(t)}{dt} \overset{\mathcal{L}}{\Rightarrow} V(s) = L\Big(sI(s) - i(0)\Big) =$$

$$= LsI(s) - Li(0)$$

If $i(0) = 0 \Rightarrow V(s) = LsI(s) \Rightarrow \dfrac{V(s)}{I(s)} = sL$

The quantity $\frac{V(S)}{I(S)}$ is called <u>IMPEDANCE</u>

Depending on the component it may or may not have the same unit of measure of the correspondent quantity in the time domain.

In the case of a resistor:

$$\frac{V(S)}{I(S)} = R \quad \Omega$$

But for other components it is not true.

We said that the trasformation is linear so we can actually transform KVL and KCL.
Basically these laws hold also in the s domain.
For instance at a node $\sum_i I_i(s) = 0$.

[If you didn't get the importance of the trasformation

Consider the differential equation:

$$a_2 \frac{d^2 v}{dt^2} + a_1 \frac{dv}{dt} + a_0 v = 0$$

If we apply the laplace transform we get:

$$a_2 \left[s\left(sV(s) - v(0)\right) - v'(0) \right] + a_1\left(sV(s) - v(0)\right) + a_0 V(s) = 0$$

$$V(s)\left(a_2 s^2 + a_1 s + a_0\right) - a_2 s v(0) - a_2 v'(0) - a_1 v(0) = 0$$

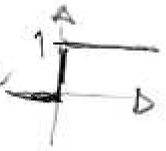
$$V(s) = \frac{a_2 s v(0) + a_2 v'(0) + a_1 v(0)}{a_2 s^2 + a_1 s + a_0}$$

$\mathcal{L}$ transforms differential equations in <u>algebraic</u> equations!!!]

Some know transformation:

$$\sin(\omega_0 t) \longrightarrow \frac{\omega_0}{s^2+\omega_0^2}$$

$$\cos(\omega_0 t) \longrightarrow \frac{s}{s^2+\omega_0^2}$$

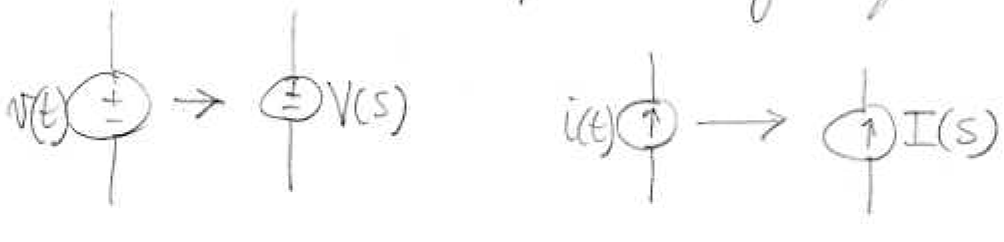
step function  $\longrightarrow \frac{1}{s}$

Dirac's delta $\delta(t) \longrightarrow 1$

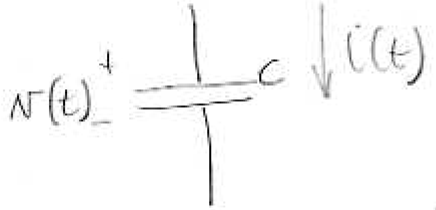
$$e^{s_0 t} \longrightarrow \frac{1}{s-s_0}$$

The method

1) Transform all sources (this is easy just use the following equivalences:

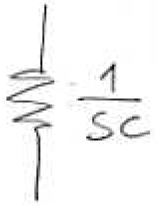


2) Transform capacitor and inductors:



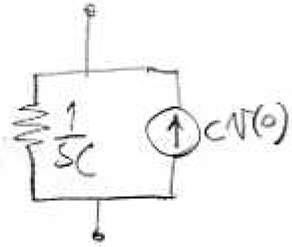
$$i(t) = C\frac{dv(t)}{dt}$$

$v(0)=0$: $I(s) = sCV(s)$

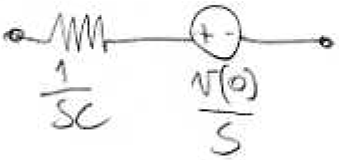


$v(0)\neq 0$:

$$I(s) = sCV(s) - v(0)C$$



$$V(s) = \frac{I(s)}{sC} + \frac{v(0)}{s}$$

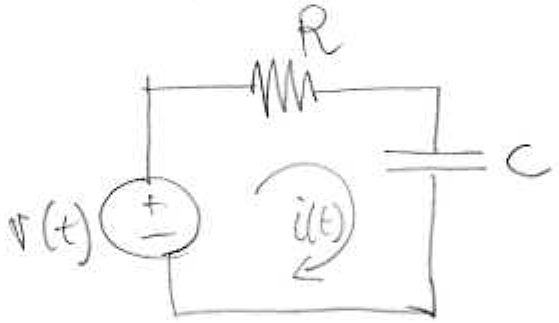


Transform all inductors (you can find the transformation by yourself)

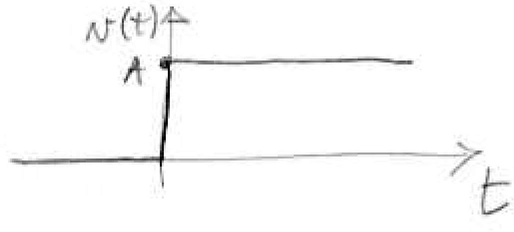
3) Use node-voltage or mesh-current analysis and find all quantities as function of s

4) Apply $\mathcal{L}^{-1}$ to find the time domain response of the circuit

Examples:

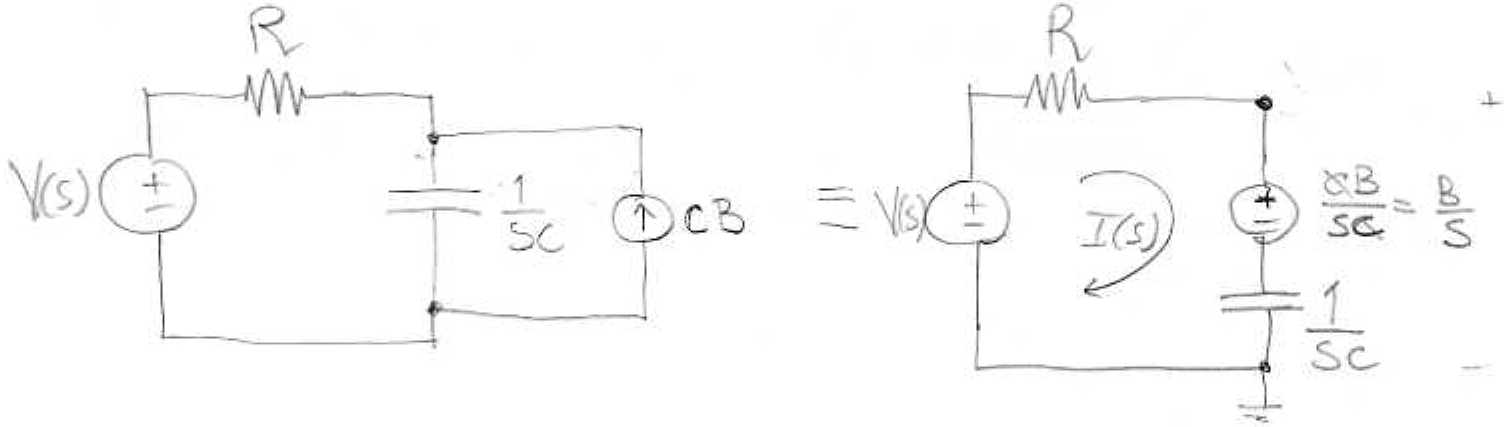


$v(t)$ is the step function:



Consider also the initial condition $v(0) = B$.

We use Laplace transform:



We want to compute the voltage across the capacitor as function of $V(s)$: Use KVL and deal with the capacitor as if it was a resistor whose value is $\frac{1}{sC}$:

$$-V(s) + RI(s) + \frac{B}{s} + \frac{I(s)}{sC} = 0$$

$$I(S) = \frac{V(S) - B/s}{R + \frac{1}{SC}} = \frac{SV(S) - B}{SR + \frac{1}{C}}$$

Now we can compute $V_c(S)$ which is not only the voltage across the impedence $\frac{1}{SC}$ but it also includes the ficticiou voltage source that we introduced to take into account the initial condition:

$$V_c(S) = I(S) \cdot \frac{1}{SC} + \frac{B}{S} = \frac{I(S) + BC}{SC} =$$

$$= \frac{\frac{sV(S) - B}{SR + \frac{1}{C}} + BC}{SC} = \frac{sV(S) - \cancel{B} + SCRB + \cancel{B}}{S^2RC + S}$$

$$= \frac{sV(S) + SCRB}{S(SRC + 1)}$$

$v(t)$ is a step function with amplitude A so (by looking at the table on page 9.10)

$V(S) = \frac{A}{S}$ so:

$$V_c(S) = \frac{A + SCRB}{S(SRC+1)}$$

To find $v(t)$ we use a method wich is call "partial fractions method".
We want to write $V_c(S)$ in the following form:

$$V_c(S) = \frac{\alpha}{S} + \frac{\beta}{(SRC+1)}$$

You can verify that:

$$\alpha = \lim_{S \to 0} V_c(S)\, S$$

$$\beta = \lim_{S \to -\frac{1}{RC}} V_c(S)\,(1+SRC)$$

So we have:

$$\alpha = A$$

$$\beta = \left.\frac{(A + SCRB)}{S}\right|_{S=-\frac{1}{RC}} = -(A-B)\,RC$$

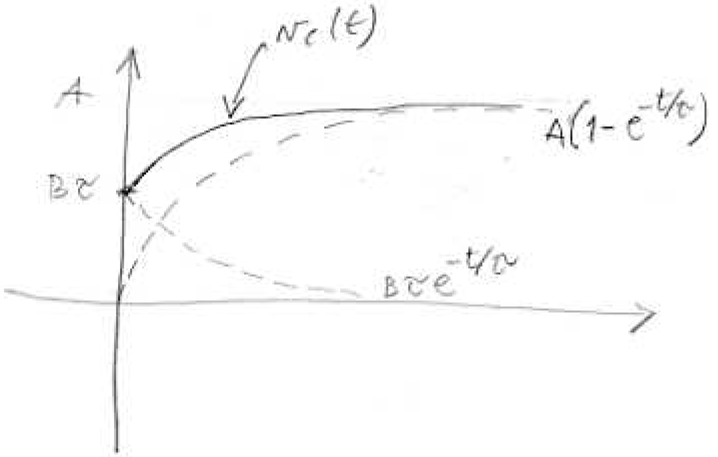
$$V_c(S) = \frac{A}{S} + \frac{(B-A)RC}{1+SRC}$$

using the table on page 9.10 we can compute $\mathcal{L}^{-1}$ just by table lookup for $t \geq 0$

$$v(t) = A + (B-A)RC\, e^{-t/\tau} = A\left(1-e^{-t/\tau}\right) + B\tau e^{-t/\tau}$$

where $\tau = RC$.

The result is very interesting. It is the sum of two things: $A(1-e^{-t/\tau})$ is the response to the step function as if the initat condition $v(0)$ was equat to zero. The other term $B\tau e^{-t/\tau}$ is the response to the initial condition.



- ANALYSIS IN THE FREQUENCY DOMAIN

We have already seen that if all sources are of the form $A\cos(\omega_0 t+\varphi)$ then also any other quantity is of the same form (pag 9.1).

Consider now the voltage source:

$$e(t) = A\cos(\omega_0 t + \varphi)$$

using complex numbers, it can be re-written as:

$$e(t) = A\,\frac{e^{j(\omega_0 t+\varphi)} + e^{-j(\omega_0 t+\varphi)}}{2} = \frac{A}{2}\left[e^{j\varphi}e^{j\omega_0 t} + \right.$$

$$\left. + \, e^{-j\varphi}e^{-j\omega_0 t}\right] = \frac{1}{2}\left[Ae^{j\varphi}e^{j\omega_0 t} + Ae^{-j\varphi}e^{-j\omega_0 t}\right] =$$

$$= \frac{1}{2}\left[E e^{j\omega_0 t} + E^* e^{-j\omega_0 t}\right]$$

where $E = Ae^{j\varphi}$ and E^* is the complex conjugate of E. E is called phasor of $e(t)$.

Now we can take the laplace transform of $e(t)$:

$$\mathcal{L}[e(t)] = \mathcal{L}\left[\frac{E e^{j\omega_0 t} + E^* e^{-j\omega_0 t}}{2}\right] =$$

$$= \frac{1}{2}\left[\frac{E}{s-j\omega_0} + \frac{E^*}{s+j\omega_0}\right] = E(s)$$

Consider now another quantity that we are interested in (maybe a voltage across a capacitor for instance), say $u(t)$. Its Laplace transform is $U(s)$.
$E(s)$ and $U(s)$ are related by a function of s that can be computed by solving the circuit. So we can write:

$$U(s) = F(s)\,E(s)$$ ($F(s)$ is called transfer function)

But we also know that $u(t) = A' \cos(\omega_0 t + \varphi')$ so its transform is:

$$U(s) = \frac{1}{2}\left[\frac{U}{s-j\omega_0} + \frac{U^*}{s+j\omega_0}\right]$$

where U is the phasor of $u(t)$. which is basically $A' e^{j\varphi'}$

If we want to compute U, we can use the same method that we used for partial fractions (pag 9.14):

$$U = \lim_{s \to j\omega_0} 2\, U(s)\,(s - j\omega_0) =$$

$$= \lim_{s \to j\omega_0} 2F(s)\frac{1}{2}\left[\frac{E}{s - j\omega_0} + \frac{E^*}{s + j\omega_0}\right](s - j\omega_0) =$$

$$= \lim_{s \to j\omega_0} F(s)\left[E + \frac{E^* \, (s - j\omega_0)}{s + j\omega_0}\right] =$$

$$= F(j\omega_0)\, \underline{E}$$

$F(j\omega_0)$ is a complex number and can hence be represented as: $F(j\omega_0) = |F(j\omega_0)|\, e^{j\measuredangle F(j\omega_0)}$

So the phasor $U = |F(j\omega_0)|\, e^{j\measuredangle F(j\omega_0)}\, \underline{E} =$

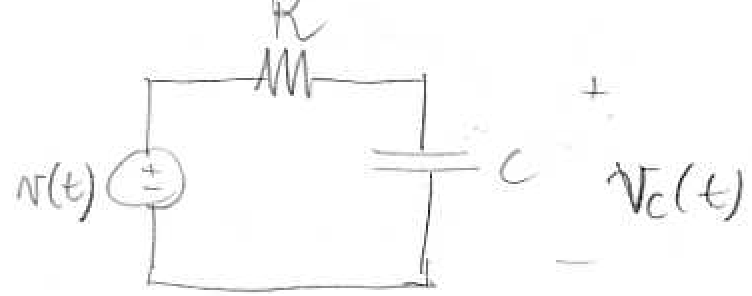
$$= |F(j\omega_0)|\, A\, e^{j(\varphi + \measuredangle F(j\omega_0))}$$

and $\quad u(t) = \underbrace{A\,|F(j\omega_0)|}_{A'} \cos(\omega_0 t + \underbrace{\varphi + \measuredangle F(j\omega_0)}_{\varphi'})$

It is important to notice that $F(j\omega_0)$ is the Fourier transform of the transfer function $\frac{u(t)}{e(t)}$ computed in ω_0.

If we leave ω_0 to be a variable, then we can understand how our circuit behaves at different frequencies.

Example:



We want to compute the frequency response of the circuit, or better the steady state sinusoidal behavior.
to do this we consider all initial conditions equal to zero (steady state)

and we compute the tranfer function

$$F(s) = \frac{V_c(s)}{V(s)}$$

$F(j\omega)$ says $\forall\ \omega$, how the circuits modifie the amplitude and phase of a sinusoidal input $v(t)$.

$$V_c(s) = V(s) \frac{1/sC}{R + 1/sC} = V(s) \frac{1}{1+sCR}$$

$$\frac{V_c(S)}{V(S)} = \frac{1}{1+SRC} = F(S)$$

$$F(j\omega) = \frac{1}{1+j\omega RC}$$

$$|F(j\omega)| = \left| \frac{(1-j\omega RC)}{(1+j\omega RC)(1-j\omega RC)} \right| = \frac{|1-j\omega RC|}{1+(\omega RC)^2} =$$

$$= \frac{\sqrt{1+(\omega RC)^2}}{1+(\omega RC)^2} = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\measuredangle F(j\omega) = \tan^{-1}\left(\frac{Im\{F(j\omega)\}}{Re\{F(j\omega)\}}\right) =$$

$$= \tan^{-1}(-\omega RC)$$

this means that if $v(t) = A\cos(\omega_0 t)$

then

$$v_c(t) = \frac{A}{\sqrt{1+(\omega_0 RC)^2}}\cos\left(\omega_0 t + \tan^{-1}(-\omega_0 RC)\right)$$

The phase sometimes could be important, but the change in the amplitude is very important because depending on the frequency the output could become very small.