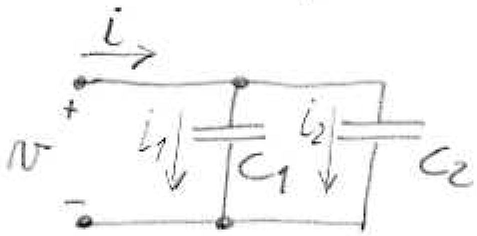
SERIES PARALLEL CONNECTIONS OF CAPACITORS AND INDUCTORS

1) Consider the <u>parallel</u> connection of two capacitors:

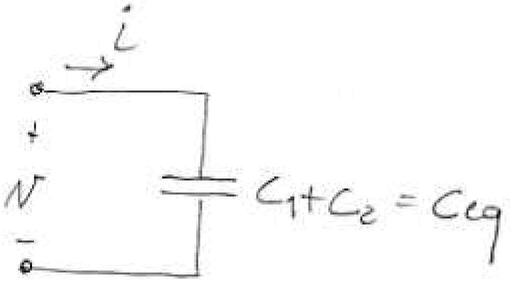


KCL: $i = i_1 + i_2$

The two capacitors are in parallel meaning that the voltage across them is the same:

$$i_1 = C_1 \frac{dv}{dt} \quad ; \quad i_2 = C_2 \frac{dv}{dt}$$

$$\Rightarrow \quad i = (C_1 + C_2)\frac{dv}{dt} \quad \equiv$$



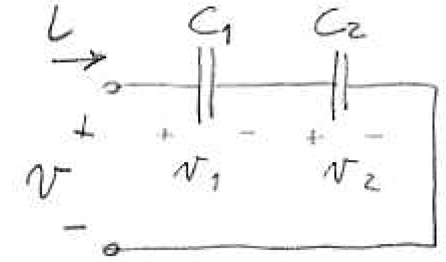
$\boxed{\text{Sum of capacitance}}$

[Define the parallel composition of capacitors:

$C_1 /\!/ C_2$ ⟵ this is an operation!

Prove that // is associative and you find a rule that applies for n capacitor in //]

Now let us consider the <u>series</u> composition of capacitors:

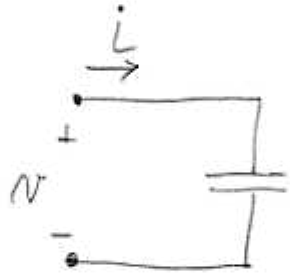


$$v_1 = \frac{1}{C_1}\int i\, dt \quad ; \quad v_2 = \frac{1}{C_2}\int i\, dt$$

The current is the same because they are in series.

KVL: $$v = v_1 + v_2 = \frac{1}{C_1}\int i\, dt + \frac{1}{C_2}\int i\, dt =$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int i\, dt = \frac{1}{C_{eq}}\int i\, dt$$

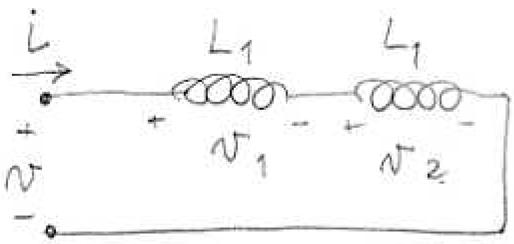


$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

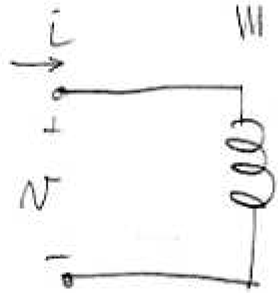
Series composition is also associative so you can find out how to compose more than 2 capacitors.

2) We now consider inductors:

SERIES



$$v_1 = L_1 \frac{di}{dt} \quad ; \quad v_2 = L_2 \frac{di}{dt}$$

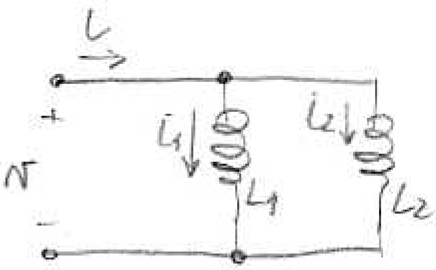


$L_{eq} = L_1 + L_2$

$$v = v_1 + v_2 = (L_1 + L_2) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

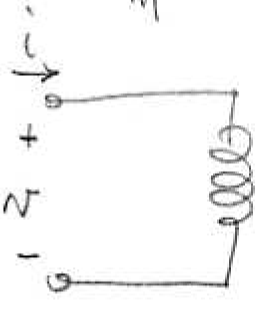
$\boxed{\text{Sum of inductance}}$

PARALLEL



$$i_1 = \frac{1}{L_1} \int v \, dt \quad ; \quad i_2 = \frac{1}{L_2} \int v \, dt$$

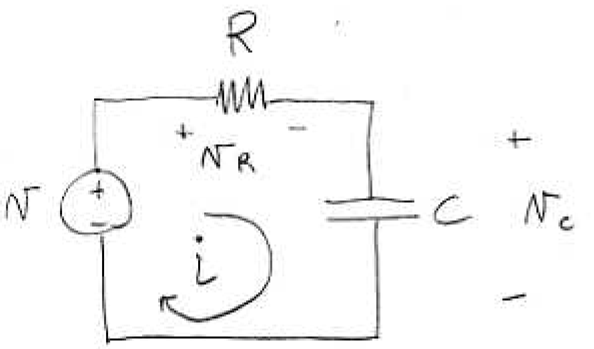
$$i = i_1 + i_2 = \left(\frac{1}{L_1} + \frac{1}{L_2} \right) \int v \, dt = \frac{1}{L_{eq}} \int v \, dt$$



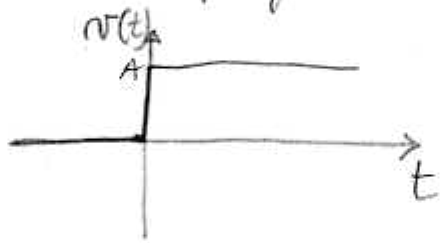
$$L_{eq} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2}}$$

$\boxed{\text{Inverse of sum of the inverses}}$

RESPONSE OF RC CIRCUITS TO STEP FUNCTION



The source is a step function:



$$v(t)=\begin{cases}0 & t\le 0\\ A & t>0\end{cases}$$

KVL: $v_c(t) = v(t) - v_R(t) = v(t) - R\,i(t) =$

$$= v(t) - RC\frac{dv_c(t)}{dt}$$

R and C are in series, the current through them is the same and the current in the capacitor is $C\frac{dv_c(t)}{dt}$

We obtain the following differential equation:

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{v(t)}{RC}$$

Since $v(t)=0$ for $t\le 0$ then we know that $v(0)=0$. To understand this better: if $v(t)=0$ then we can replace the voltage source with a short circuit

Now we want to write the equation governing this circuit:

$$i(t) = C\frac{dv_c(t)}{dt} = \frac{v_R(t)}{R} = -\frac{v_c(t)}{R}$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = 0$$

We want to understand the following: given an initial condition at $t=-\infty$ what would be $v_c(0)$, the voltage across the capacitor at time $t=0$?

This problem is equivalent to the following: given an initial condition at time $t=0$, what would be $v_c(\infty)$, the voltage across the capacitor at time $t=\infty$?

$v_c(0) = A \neq 0$ $\qquad \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = 0$

consider the solution $v_c(t) = \gamma e^{\alpha t} + \beta$

1) $\gamma\alpha e^{\alpha t} + \frac{1}{RC}\gamma e^{\alpha t} + \frac{1}{RC}\beta = 0$

2) $v_c(0) = A \Rightarrow \gamma e^{\alpha 0} + \beta = \gamma + \beta = A \Rightarrow \beta = A - \gamma$

from 1: $\gamma\alpha e^{\alpha t} + \frac{1}{RC}\gamma e^{\alpha t} + \frac{1}{RC}(A-\gamma) = 0$

$$\gamma e^{\alpha t}\left(\alpha + \frac{1}{RC}\right) + \frac{1}{RC}(A-\gamma) = 0$$

$$\Rightarrow \gamma = A \Rightarrow A e^{\alpha t}\left(\alpha + \frac{1}{RC}\right) = 0 \Rightarrow \alpha = -\frac{1}{RC}$$

$\beta = 0$

$v_c(t) = A e^{-t/RC}$

now $\lim_{t\to\infty} v_c(t) = 0$ no matter what A is!

□

So in our original problem we have

$\boxed{v_c(0) = 0}$. ← (valid for $t>0$)

$RC\frac{dv_c(t)}{dt} + v_c(t) = A$ we consider the solution

$$v_c(t) = \gamma e^{\alpha t} + \beta$$

$$RC\gamma\alpha e^{\alpha t} + \gamma e^{\alpha t} + \beta = A \Rightarrow \beta = A$$

$$RC\gamma\alpha \cancel{e^{\alpha t}} + \gamma \cancel{e^{\alpha t}} = 0 \Rightarrow RC\gamma\alpha + \gamma = 0 \overset{\gamma\neq 0}{\Rightarrow} \alpha = -\frac{1}{RC}$$

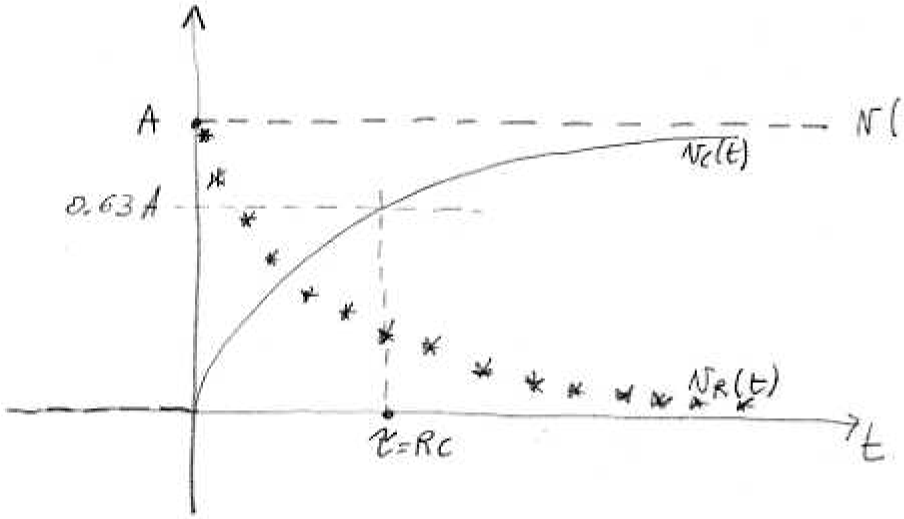
$$v_c(0) = 0 \Rightarrow \gamma e^{\alpha 0} + A = 0 \Rightarrow \gamma = -A$$

the voltag then has the following expression:

$$(t>0) \quad v_c(t) = A - Ae^{-\frac{t}{RC}} = A\left(1-e^{-\frac{t}{RC}}\right)$$

Consequently:

$$v_R(t) = v(t) - v_c(t) = A - A + Ae^{-t/RC} = $$
$$= Ae^{-t/RC}$$



for $t = \tau = RC$

$$v_c(\tau) = A\left(1 - e^{-1}\right) = 0.63\,A$$

the voltage reaches 63% of the input voltage

We will now consider the behaviour of $v_R(t)$ and $v_c(t)$ for extreme values of RC

Using Taylor expansion:

$$v_c(t) = A\left(1-\left(1-\frac{1t}{\tau}+\frac{1t^2}{2\tau^2}+O(\tau^{-3})\right)\right) =$$

$$= A\left(\frac{t}{\tau}+O(\tau^{-2})\right)$$

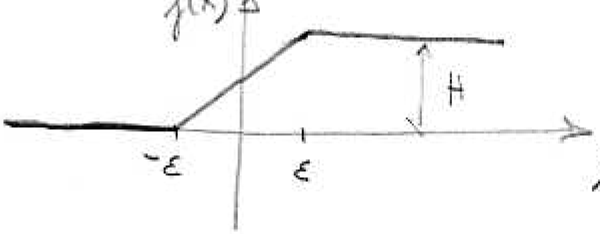
If τ is very very big then $O(\tau^{-2})$ can be neglected and we obtain

$$v_c(t) \simeq \frac{At}{\tau} \propto \int v(t)\,dt$$

The voltage across the capacitor is approximately equal to the integral of the input (if t becomes big though then $O(\tau^{-2})$ cannot be neglected anymore)

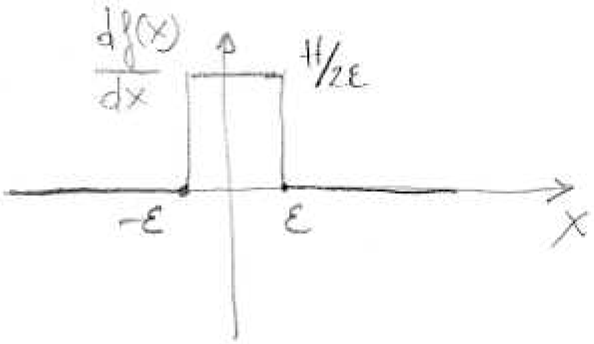
Before analyzing the behaviour of $v_R(t)$ we introduce a special function.

[consider a $f(x)$ like the following



$$f(x) = \begin{cases} 0 & x \leq -\varepsilon \\ \frac{H}{2\varepsilon}x + \frac{H}{2} & -\varepsilon < x < \varepsilon \\ H & x \geq \varepsilon \end{cases}$$

Its derivative is:



$$g(x) = \frac{df(x)}{dt} = \begin{cases} 0 & x \le -\varepsilon \\ H/2\varepsilon & -\varepsilon < x < \varepsilon \\ 0 & x \ge \varepsilon \end{cases}$$

also $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = H$

$\lim_{\varepsilon \to 0} f(x) = \begin{cases} 0 & x \le 0 \\ H & x > 0 \end{cases}$ is the step function

$\lim_{\varepsilon \to 0} g(x) = \delta(x)$ its a new function called Dirac's delta function

this function is concentrated at the origin on its value is ∞ but the integral is finite!!
So the derivative of the step function is the Dirac's delta function.
You can prove also that:

$$\lim_{x \to 0} \frac{\sin x}{x} = \delta(x)$$

$$\lim_{\sigma \to 0} \frac{1}{2\sqrt{\pi\sigma}} e^{-\frac{x^2}{4\sigma}} = \delta(x)$$

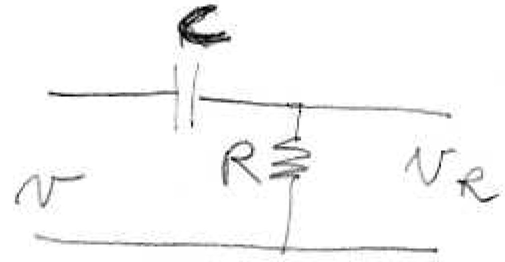
$$V_R(t) = Ae^{-t/RC} = A\sqrt{\pi RC}\left(\frac{1}{\sqrt{\pi RC}}e^{-\frac{t}{RC}}\right)$$

for RC very very small, this approximate $\delta(t)$ whis is the derivative of the input !

the problem with this circuits is that RC also affects the amplitude of the output

Summary:

approximate an integration

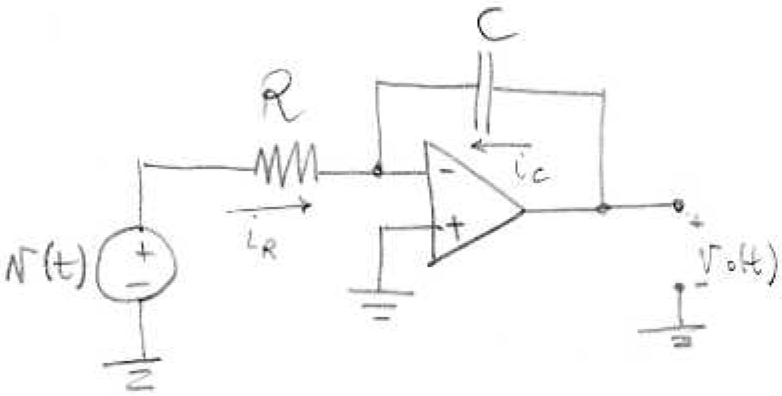


approximate a derivative

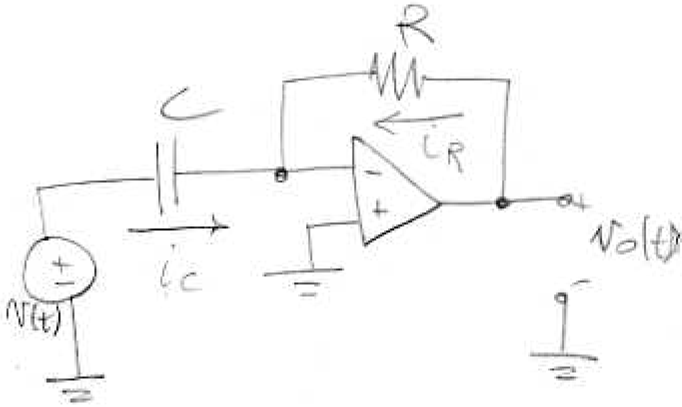
INTEGRATOR / DIFFERENTIATOR

Problems:

given $v(t)$, compute $\frac{dv(t)}{dt}$ and $\int v(t)\,dt$



1



2

1 $i_R = -i_C$

$$i_R = \frac{v(t)}{R} \Rightarrow v_o(t) = \frac{1}{C}\int_{t_0}^{t} i_C(t)\,dt + v_c(t_0) = -\frac{1}{RC}\int v(t)\,dt + v_c(t_0)$$

capacitor initial condition

This is an integrator

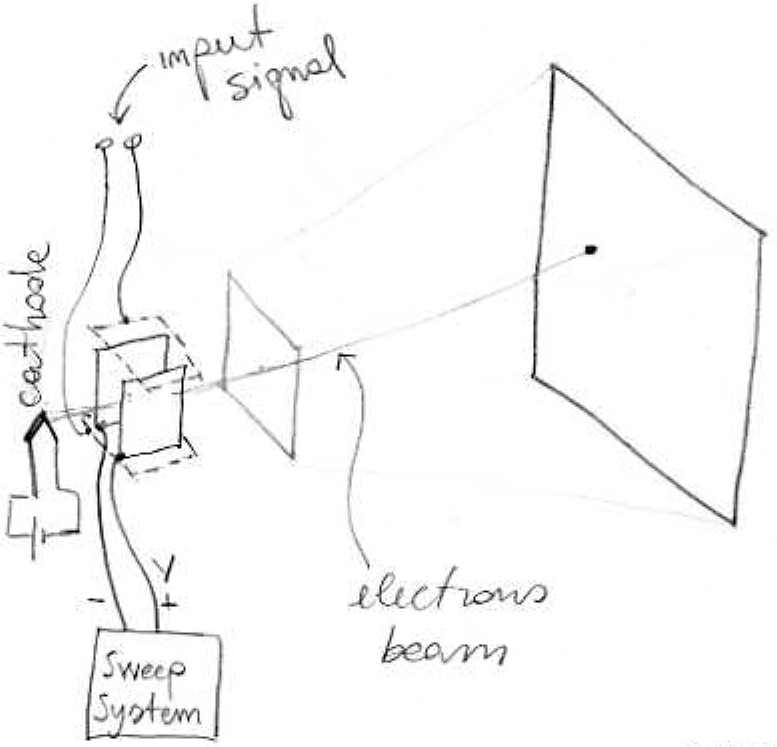
2 $i_C = -i_R$

$$i_C = C\,\frac{dv(t)}{dt} \Rightarrow v_o(t) = R\,i_R = -RC\,\frac{dv(t)}{dt}$$

This is a differentiator

Application of the integrator

OSCILLOSCOPE



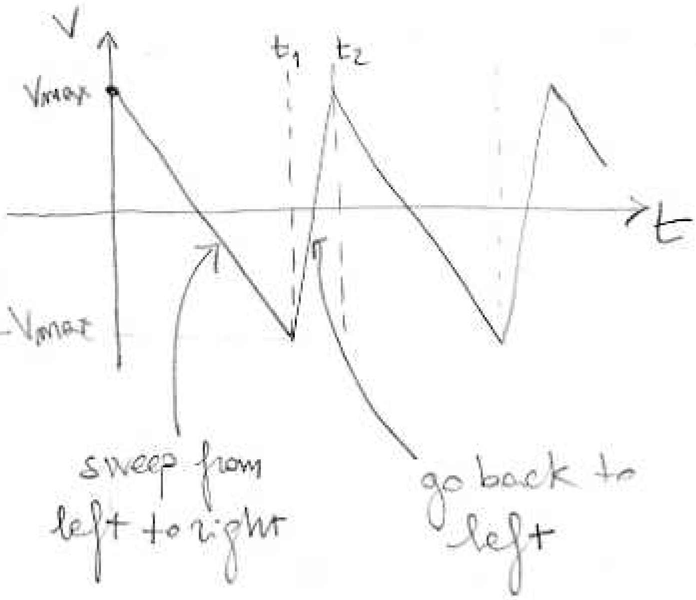
a cathode generates an electron beam. When the beam hit the surface of the tube we see a small point.

If you can generate a point, then you can move it very fast an have the illusion of a fixed picture

the input signal is connected (after amplificaton an other small conditioning) to the vertical deflecton plates. a voltage acros the plates can deviate the beam because it generates an electric field. An electron

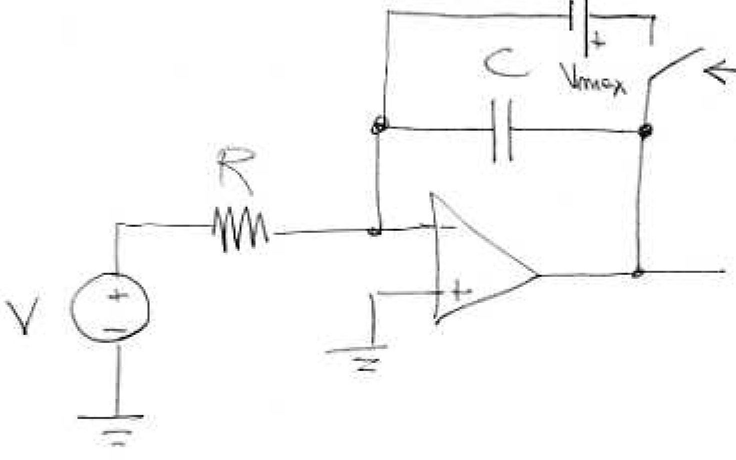
then is subject to a force $-qE$ which will change its direction.

the problem we have is that the scope is horizontally limited. We have to sweep the beam from left to right and then go back. What we want to do is the following:



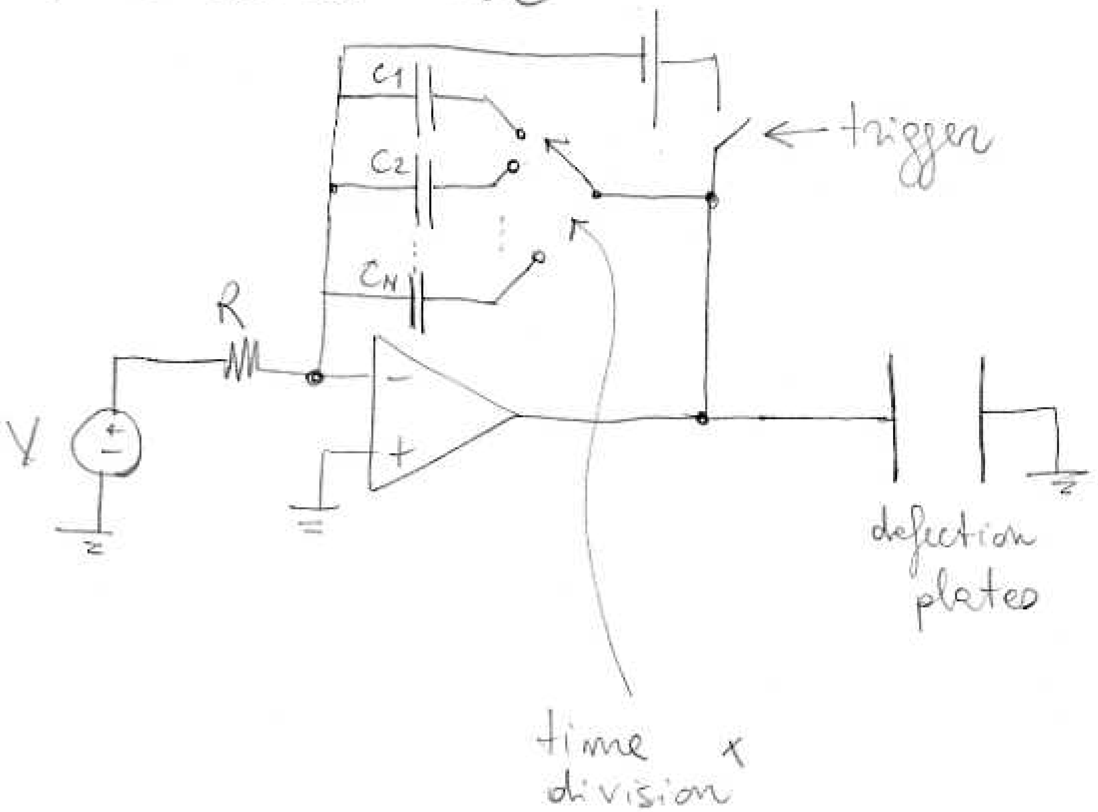
ideally we would like to go back to left position in no time but this is impossible. It is important then to block the beam between t_1 and t_2 so that we don't see it going back on the screen.

We can use a circuit like this:



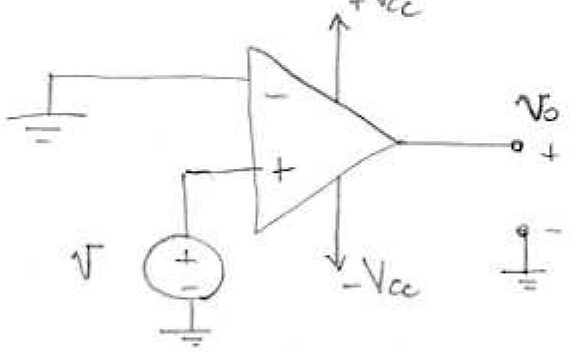
this circuit will integrate the input constant V so the output will be a negative ramp!

When the output voltage is $-V_{max}$ we trigger the switch that will set the output voltage to V_{max} again.
The switch will force an initial condition on the capacitor.
The product RC defines the slope of the linear function. We can change this product to make the sweeping faster or slower.
When you change time scale on your oscilloscope you are basically changin C and hence RC

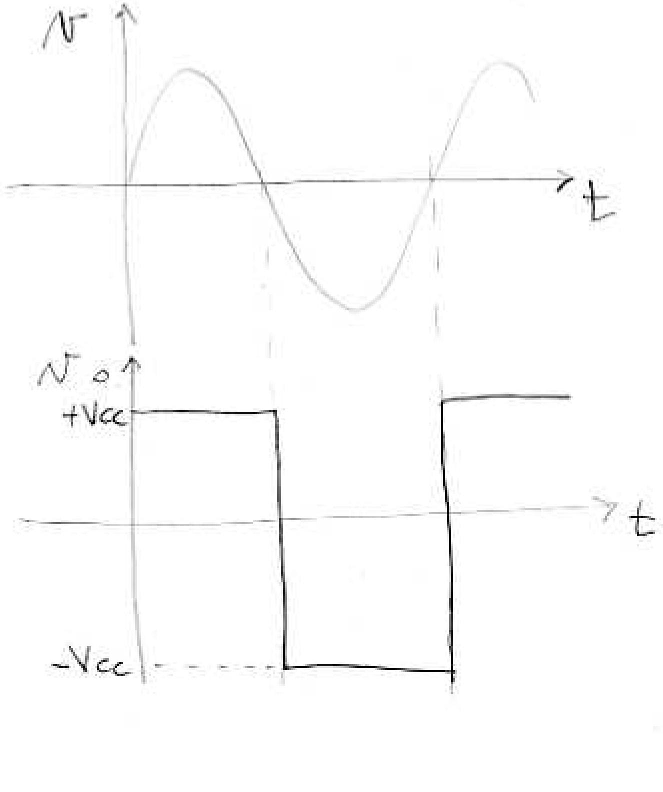


COMPARATOR

We now consider an operational amplifier with its power supply. Usually, op-amps require a dual power supply ($-V_{cc}, +V_{cc}$). The output voltage cannot exceed the power supply voltage meaning that $-V_{cc} \le V_o \le V_{cc}$. Consider the following circuit:



If $v > 0$, since $v_o = k v$ and k is ideally ∞, the output is $+V_{cc}$. If $v < 0$ the output is $-V_{cc}$ for the same reason:



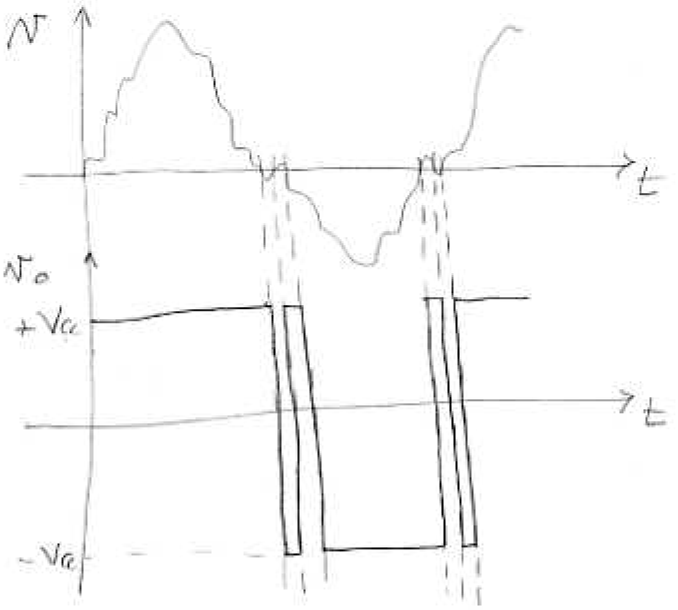
each transition of v_o

$+V_{cc} \to -V_{cc}$

$-V_{cc} \to V_{cc}$

represents a zero crossing of the input

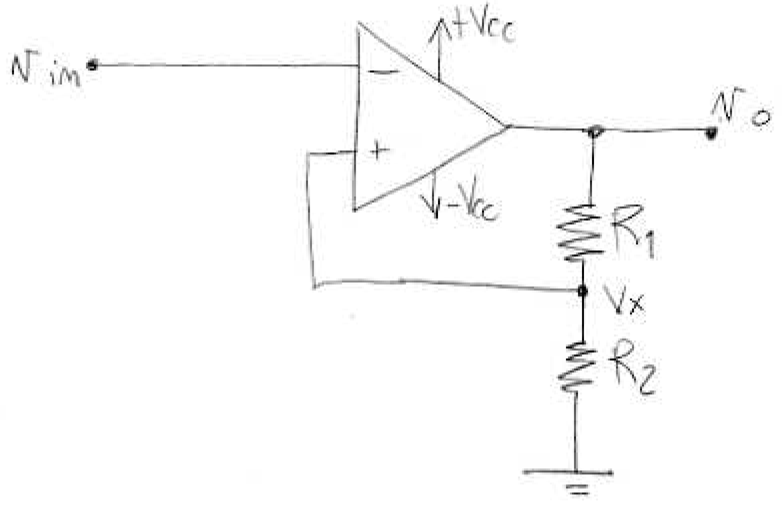
The problem with this circuit is that it's very sensitive to noise



Noise can cause spurious oscillation. We need to "enlarge" the zero axis somehow.

We can do that by changing the value of the voltage that the input signal is compared against.

shmitt's trigger:



[Note: comparators are non linear circuit so you cannot apply general circuits analysis methods]

If $v_{in} > v_x \Rightarrow v_o = -V_{cc}$

If $v_{in} < v_x \Rightarrow v_o = +V_{cc}$

The keypoint here is that v_x changes if v_o changes:

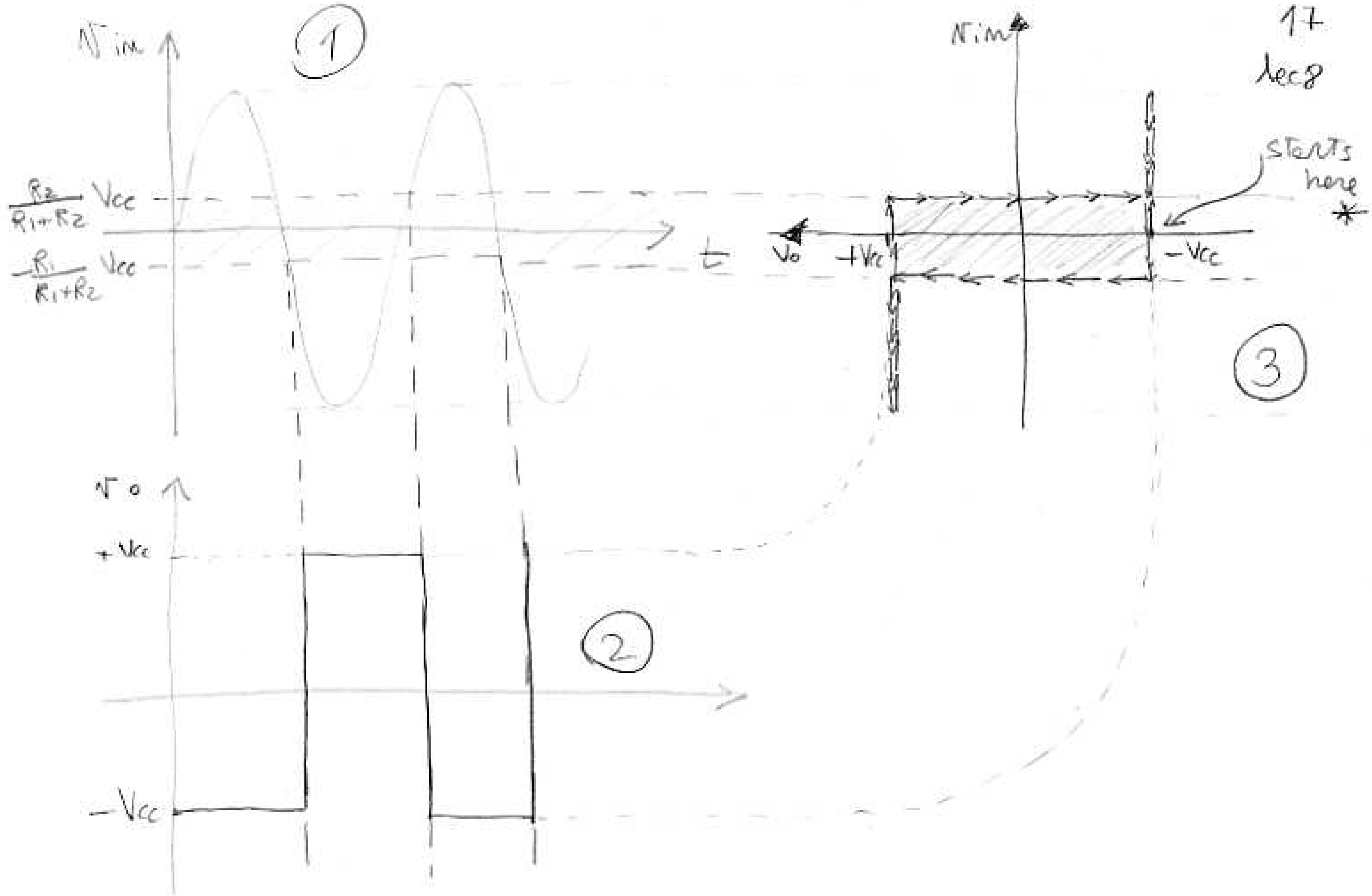
$$v_x = -\frac{R_2}{R_1+R_2} V_{cc} \qquad \text{if } v_o = -V_{cc}$$

$$v_x = \frac{R_2}{R_1+R_2} V_{cc} \qquad \text{if } v_o = +V_{cc}$$

Let's start from the initial condition $v_{in} > 0$ and $v_o = -V_{cc}$. The transition $-V_{cc} \to +V_{cc}$ only happens if $v_{in} < v_x$ and since $v_o = -V_{cc}$, $v_x = -\frac{R_2}{R_1+R_2} V_{cc}$. When $v_{in} < -\frac{R_2}{R_1+R_2} V_{cc}$ the output goes to $+V_{cc}$.

Now a new transition $+V_{cc} \to -V_{cc}$ only happens if $v_{in} > v_x = \frac{R_2}{R_1+R_2} V_{cc}$

So basically if something happens in the horizontal strip $-\frac{R_1}{R_1+R_2} V_{cc} < v_{in} < \frac{R_2}{R_1+R_2} V_{cc}$ the output v_o doesn't change.



This plot shows the histeresis. (1) is the input signal, (2) is the output signal and (3) shows v_o an v_{in} on the same graph. Starting from * you can follow the arrows to see how the output changes as function of the input. The shaded area in the plots is the common way of representing the hysteresis phenomenon.

If we want only pulses to represent zerocrossing, then we can just take derivative of the comparator output. We just build the series of the two circuits: